Proceedings of the 1998 IEEE
International Conference on Control Applications
Trieste, Italy 1-4 September 1998

FA02

Navigation System Design using Time-Varying Complementary Filters *

A. Pascoal†‡ I. Kaminer† P. Oliveira ‡

Abstract

The paper introduces a new methodology for the design of time-varying complementary filters and describes its application to the problem of estimating the position and velocity of autonomous vehicles.

1 Introduction

Currently, there is considerable interest in the development of navigation systems to provide robotic vehicles with the capability to perform complex missions autonomously. See [1, 10, 13] and the references therein for in depth presentations of navigation systems for aircraft and [5, 8, 14] for an overview of similar systems and related research issues in the underwater robotics area.

Traditionally, navigation system design is done in a stochastic setting using Kalman-Bucy filtering theory [2]. In the case of nonlinear systems, design solutions are usually sought by resorting to Extended Kalman filtering techniques [2]. The stochastic setting requires a complete characterization of process and observation noises, a task that may be difficult, costly, or not suited to the problem at hand. This issue is argued at great length in [3], who points out that in a great number of practical applications the filter design process is entirely dominated by constraints that are naturally imposed by the sensor bandwidths. In this case, a design method that explicitly addresses the problem of merging information provided by a given sensor suite over distinct, yet complementary frequency regions is warranted.

Complementary filters have been developed to address this issue explicitly. See for example [3, 10] for a concise introduction to complementary filters and their applications. In the linear time-invariant setting, filter design is ultimately reduced to the problem of decomposing an identity operator into stable low and high pass transfer functions that operate on complementary sensor information. The bandwidth of the low pass transfer function becomes a tuning parameter aimed at matching the physical characteristics of the "low frequency" sensor. Therefore, the emphasis is shifted from a stochastic to a deterministic framework, where the main objective is to shape the filter closed-transfer functions.

This paper extends complementary filter design and analysis techniques to a time-varying setting and offers a solution to the problem of estimating the linear position and velocity of a vehicle using time-varying complementary filters. The time-dependence is imposed by the fact that some of the sensors provide measurements in inertial coordinates, while other measurements are naturally expressed in body axis. To merge the information from both types of sensors - while being able to compensate for sensor biases - requires that the rotation matrix from inertial to body axis be explicitly included in the navigation filters. The resulting filters are bilinear and time-varying, but the time dependence is well structured. By exploiting this structure, the problem of filter design and analysis can be converted into that of determining the feasibility of a set of Linear Matrix Inequalities (LMIs) that arise in the theory of linear differential inclusions [4]. As a consequence, the stability of the resulting filters as well as their "frequency-like" performance can be assessed using efficient numerical analysis tools that borrow from convex optimization techniques [4, 9].

The paper is organized as follows. Section 2 reviews some basic mathematical background on linear time-varying systems, induced operator norms, and polytopic systems. Section 3 sets the motivation for the sections that follow: a simple filtering problem is formulated, and its solution in terms of complementary linear time-invariant filters is described. Section 4 describes the navigation problem addressed in this paper and formulates it mathematically in terms of an equivalent time-varying filter design problem. Section 5 provides the main theoretical tools for linear time-varying filter design and analysis using the theory of linear matrix inequalities. Finally, Section 6 describes a practical algorithm for complementary filter design and illustrates the performance of the new filtering structure in simulation.

Due to space limitations the proofs of all the results are omitted. The reader is referred to [12] for complete details.

2 Mathematical background

The objective of this section is twofold: i) describing numerical algorithms for the computation of induced operator norms for linear time-invariant and time-varying polytopic systems, and ii) introducing the key concepts of low and high pass linear systems in a linear time-varying (LTV) setting. See [15] for the notation and basic results.

---

*Work supported in part by the Portuguese Programme PRAXIS XXI under contract PRAXIS/2/2.1/TPAR/2042/95 and by the US Office of Naval Research (ONR). The first author benefited from a NATO Fellowship during his 1997-98 sabbatical at the NPS.

†Department of Aeronautics and Astronautics, Naval Postgraduate School, Monterey, CA 93943, USA.

‡Department of Electrical Engineering and Institute for Systems and Robotics and , Instituto Superior Técnico, Av. Rovisco Pais, 1096 Lisboa Codex, Portugal.

0-7803-4104-X/98/$10.00 ©1998 IEEE

2.1 Linear polytopic systems. Induced operator norms.

The symbol L_2 denotes the Hilbert space of Lebesgue measurable functions from $\mathcal{R}_+$ to $\mathcal{R}^p$ endowed with the usual operator norm, while L_{2e} denotes the corresponding extended space. An input-output system $\mathcal{G}$ is identified with an operator $\mathcal{G} : L_{2e} \to L_{2e}$. A causal system $\mathcal{G} : L_2 \to L_2$ is *(finite-gain) stable* if the L_2 *induced operator norm* $||\mathcal{G}||_{2,i}$ (abbv. $||\mathcal{G}||$) is finite. In what follows we restrict ourselves to the class of LTV systems $\mathcal{G}$ with finite-dimensional state-space realizations $\Sigma_{\mathcal{G}} := \{A(t), B(t), C(t)\}$ of bounded, piece-wise continuous matrix functions of time. Often, we will use the same symbol $\mathcal{G}$ to denote both an LTV system and its particular realization $\Sigma_{\mathcal{G}}$, as the meaning will become clear from the context. We assume the reader is familiar with the concept of exponential stability of LTV systems. To simplify the exposition, we will henceforth refer to an exponentially stable system as *internally stable*, while a (finite-gain) stable system will be simply called stable. If $\mathcal{G} : L_{2e} \to L_{2e}$ has an internally stable realization, then $\mathcal{G}$ defines a stable operator from $L_2 \to L_2$.

Let $\mathcal{G}$ be a stable linear time invariant (LTI) system with a minimal realization $\Sigma_{\mathcal{G}} := \{A, B, C\}$, and let $G(s) = C(sI - A)^{-1}B$ denote the corresponding transfer matrix. Then, the induced operator norm $||\mathcal{G}||$ equals the $\mathcal{H}_\infty$ norm of G, denoted $||G||_\infty$, where $||G||_\infty := \sup\{\sigma_{max}(G^T(-j\omega)G(j\omega)) : \omega \in \mathcal{R}\}$ and $\sigma_{max}(.)$ denotes the maximum singular value of a matrix. Given a positive integer $\gamma > 0$, then $||\mathcal{G}|| < \gamma$ if and only if there exists a positive definite matrix P that satisfies the matrix inequality [4]

$$\begin{bmatrix} A^T P + PA + C^T C & PB \\ B^T P & -\gamma^2 I \end{bmatrix} < 0 \qquad (1)$$

The above matrix inequalities are linear matrix inequalities (LMIs) in the matrix variable P. Checking for the existence of $P > 0$ is easily done by resorting to widely available numerical algorithms [9]. In this paper, we will also deal with linear time-varying systems with realizations $\{A(t), B(t), C(t)\} \in \Omega := \mathbf{Co}\{\{A_1, B_1, C_1\}, ..., \{A_L, B_L, C_L\}\}$ where

$$\mathbf{Co}S := \{\sum_{i=1}^{L} \lambda_i \mathcal{A}_i | \mathcal{A}_i \in S, \lambda_1 + ... + \lambda_L = 1\}$$

is the convex hull of the set $S := \{\mathcal{A}_1, ..., \mathcal{A}_L\}$. These systems are usually referred to in the literature as *polytopic differential inclusions* [4]. It can be shown that given a polytopic system $\mathcal{G}$, then $||\mathcal{G}|| < \gamma$ if there exists a positive definite matrix P such that

$$\begin{bmatrix} A_i^T P + PA_i + C_i^T C_i & PB_i \\ B_i^T P & -\gamma^2 I \end{bmatrix} < 0; i = 1, ..., L (2)$$

Again, checking that such a P exists can be done quite efficiently using highly efficient numerical algorithms.

2.2 Low and high pass filters.

The concept of low pass and high pass filters is well understood in the case of linear time-invariant systems. We now extend these concepts to the class of linear time-varying systems. The new concepts will play a major role in assessing the performance of the linear time-varying complementary filters that will be introduced later.

Definition. Low pass property. Let $\mathcal{G}$ be a linear, internally stable time-varying system and let $\mathcal{W}_\omega^n$ be a low-pass, linear time-invariant Chebyschev filter of order n and cutoff frequency ω. The system $\mathcal{G}$ is said to satisfy a low pass property with indices (ϵ, n) over $[0, \omega_c]$ if $||(\mathcal{G} - I) W_{\omega_c}^n|| < \epsilon$

Definition. Low pass filter with bandwidth ω_c. A linear, internally stable time-varying system $\mathcal{G}$ is said to be an (ϵ, n) *low pass filter* with bandwidth ω_c if

- $\lim_{\omega \to 0} ||(\mathcal{G} - I) W_\omega^n||$ is well defined and equals 0.
- $\omega_c := \sup\{\omega : ||(\mathcal{G} - I) W_\omega^n|| < \epsilon\}$, i.e. $\mathcal{G}$ satisfies a low pass property with indices (ϵ, n) over $[0, \omega]$ for all $\omega \in [0, \omega_c)$ but fails to satisfy that property whenever $\omega \geq \omega_c$.
- For every $\delta > 0$, there exists $\omega^* = \omega^*(\delta)$ such that $||\mathcal{G}(I - W_\omega^n)|| < \delta$ for $\omega > \omega^*$.

Definition. High Pass Filter with break frequency ω_c. A linear, internally stable time-varying system $\mathcal{G}$ is said to be an (ϵ, n) high pass filter with break frequency ω_c if $(I - \mathcal{G})$ is an (ϵ, n) low pass filter with bandwidth ω_c.

The conditions in the definition of low pass filters generalize the following facts that are obvious in the linear time-invariant case: i) the filter must provide a gain equal to one at zero frequency, ii) there is a finite band of frequencies over which the system behaviour replicates very closely that of an identity operator, and iii) the system gain rolls-off to zero at high frequency. Notice the role played by the weighting operator $\mathcal{W}_\omega^n$, which was arbitrarily selected as a Chebyschev filter. In practice, the order of the filter can be made arbitrarily large so that the filter will effectively select the "low frequency components" of the input signal.

3 Complementary filters: basic concepts and definitions.

Complementary filters arise naturally in the context of signal estimation based on measurements provided by sensors over distinct, yet complementary regions of frequency. Brown [3] was the first author to stress the importance of complementary filters in navigation system design. Since then, this subject has been studied in a number of publication that address theoretical as well as practical implementation issues; see for example [1, 10, 11, 13] and the references therein. The key ideas in complementary filtering are very intuitive, and can be simply introduced by referring to the example of Figure 1. The figure captures the practical situation where it is required to estimate the heading ψ of a vehicle based on measurements r_m and ψ_m of $r = \dot{\psi}$ and ψ respectively, provided by a rate gyro and a fluxgate compass. The measurements are corrupted by disturbances r_d and ψ_d. Let $\psi(s)$ and $r(s)$ denote the Laplace Transforms of ψ and r, respectively. Then, for every $k > 0$, $\psi(s)$ admits the stable decomposition

$$\psi(s) = \frac{s+k}{s+k}\, \psi(s) = T_1(s)\psi(s) + T_2(s)\psi(s), \qquad (3)$$

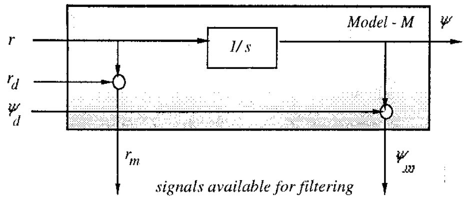

Figure 1: Process model.

where $T_1(s) = k/(s+k)$ and $T_2(s) = s/(s+k)$ satisfy the equality $T_1(s) + T_2(s) = I$. Using the relationship $r(s) = s\psi(s)$, it follows from the above equations that $\psi(s) = F_\psi(s)\psi(s) + F_r(s)r(s)$ where $F_\psi(s) = T_1(s) = k/(s+k)$ and $F_r(s) = 1/(s+k)$. This suggests a complementary filter with the structure $\hat{\psi} = \mathcal{F}_\psi \psi_m + \mathcal{F}_r r_m$ where $\mathcal{F}_\psi$ and $\mathcal{F}_r$ are linear time-invariant operators with transfer functions $F_\psi(s)$ and $F_r(s)$, respectively. Clearly, the filter admits the state space realization $\dot{\hat{\psi}} = -k\hat{\psi} + k\psi_m + r_m = r_m + k(\psi_m - \hat{\psi})$ that is represented in figure 2. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ denote linear time-invariant operators with transfer functions $T_1(s)$ and $T_2(s)$, respectively. Simple computations show that $\hat{\psi} = (\mathcal{T}_1 + \mathcal{T}_2)\psi + \mathcal{F}_\psi \psi_d + \mathcal{F}_r r_d$, that is, the estimate $\hat{\psi}$ consists of an undistorted copy $(\mathcal{T}_1 + \mathcal{T}_2)\psi = \psi$ of the original signal ψ, together with corrupting terms that depend on the measurement disturbances ψ_d and r_d. Notice the following important properties:

- $T_1(s)$ is low-pass: the filter relies on the information provided by the compass at low frequency only.
- $T_2(s) = I - T_1(s)$: the filter blends the information provided by the compass in the low frequency region with that available from the rate gyro in the complementary region.
- the break frequency is simply determined by the choice of the parameter k.

The frequency decomposition induced by the complementary filter structure holds the key to its practical success, since it mimics the natural frequency decomposition induced by the physical nature of the sensors themselves: compasses provide reliable information at low frequency only, whereas rate gyros exhibit biases and drift phenomena in the same frequency region and are therefore useful at higher frequencies. Complementary filter design is then reduced to the computation of the gain k so as to meet a target break frequency that is entirely dictated by the physical characteristics of the sensors. From this point of view, the emphasis is shifted from a stochastic framework - that relies heavily on a correct description of process and measurement noise [3] and the minimization of filter errors - to a deterministic framework that aims at shaping the filter closed-transfer functions.

As convincingly argued in [3], the latter approach is best suited to tackle a large number of practical situations where the characterization of process and measurement disturbances in a stochastic context does not fit the problem at hand, the filter design process being entirely dominated by the constraints imposed by

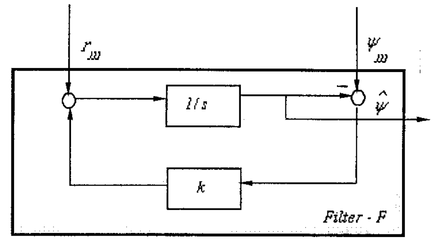

Figure 2: Complementary filter.

sensor bandwidths. Once this set-up is adopted, however, one is free to adopt any efficient design method, the design parameters being simply viewed as "tuning knobs" to shape the characteristics of the closed loop operators. In this context, filter design can be done using H_2 or H_∞ design techniques [2, 6, 7, 11]. Filter analysis is easily carried out in the frequency domain using Bode plots. In the simple case described here, the underlying process model can be written as

$$\begin{cases} \dot{\psi} & = r_m - r_d \\ \psi_m & = \psi + \psi_d \end{cases} \tag{4}$$

where r_d and ψ_d play the roles of process and measurement disturbances, respectively. Notice the important fact that ψ_m (the measured value of ψ) is an input to the system. In an H_2 setting, the objective is to minimize the estimation error $\psi - \hat{\psi}$ for given values of the covariances of ψ_d and r_d. The optimal solution to this problem has the complementary filter structure described before. The covariances of ψ_d and r_d are simply viewed as design parameters to vary the break frequency.

In practice, the simple complementary structure described above can be modified to meet additional constraints. For example, to achieve steady state rejection of the rate gyro bias, the filter must be augmented with an integrator to obtain a new complementary filter structure. See [12] for details. In view of the discussion above, we henceforth adopt a deterministic framework for complementary filter design and analysis where the objective is to shape the filter transfer functions to obtain desired bandwidths. Furthermore, in preparation for the sections that follow, it is convenient to formally introduce the definition of a complementary filter for the underlying process model (4) (with $r_d = \psi_d = 0$) in a state-space framework, see figure 1.

Definition. (r, ψ) **Complementary Filter**. Consider the process model

$$\mathcal{M}_{\psi r} := \begin{cases} \dot{\psi} & = r \\ \psi_m & = \psi \\ r_m & = r \end{cases} \tag{5}$$

and a filter $\mathcal{F}$ with realization

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B_r r_m + B_\psi \psi_m \\ \hat{\psi} &= C\mathbf{x} \end{aligned}$$

Then, $\mathcal{F}$ is said to be a complementary filter for $\mathcal{M}_{\psi r}$ if

- $\mathcal{F}$ it is internally stable
- For every any initial conditions $\psi(0)$ and $\mathbf{x}(0)$ $lim_{t\to\infty}\{\psi(t) - \hat{\psi}(t)\} = 0$.

- $\mathcal{F}$ satisfies a bias rejection property, that is, $lim_{t\to\infty}\hat{\psi} = 0$ when $\psi_m = 0$ and r_m is an arbitrary constant.
- The operator $\mathcal{F}_\psi : \psi_m \to \hat{\psi}$ is a finite bandwidth low pass filter.

4 Navigation system design: problem formulation

This section describes the navigation problem that is the main focus of the paper and formulates it mathematically in terms of an equivalent filter design problem. For the sake of clarity, we first introduce some basic notation and summarize the kinematic equations for a general vehicle.

4.1 Notation. Vehicle kinematics: a summary.

Let $\{\mathcal{I}\}$ be a reference frame, and let $\{\mathcal{B}\}$ denote a body-fixed frame that moves with the vehicle. The following notation is required: $\mathbf{p} = [x\ y\ z]^T$ - position of the origin of $\{\mathcal{B}\}$ measured in $\{\mathcal{I}\}$; ${}^I\mathbf{v} = [\dot{x}\ \dot{y}\ \dot{z}]^T$ - linear velocity of the origin of $\{\mathcal{B}\}$ measured in $\{\mathcal{I}\}$; $\mathbf{v} = [u\ v\ w]^T$ - linear velocity of the origin of $\{\mathcal{B}\}$ with respect to $\{\mathcal{I}\}$, resolved in $\{\mathcal{B}\}$; $\boldsymbol{\omega} = [p\ q\ r]^T$ - angular velocity of $\{\mathcal{B}\}$ with respect to $\{\mathcal{I}\}$, resolved in $\{\mathcal{B}\}$; $\boldsymbol{\lambda} = [\phi\ \theta\ \psi]^T$ - vector of roll, pitch, and yaw angles that parametrize locally the orientation of frame $\{\mathcal{B}\}$ with respect to $\{\mathcal{I}\}$.

Given two frames $\{\mathcal{A}\}$ and $\{\mathcal{B}\}$, ${}^A_B\mathcal{R}$ denotes the rotation matrix from $\{\mathcal{B}\}$ to $\{\mathcal{A}\}$. In particular, ${}^I_B\mathcal{R}$ (abbreviated $\mathcal{R}$) is the rotation matrix from $\{\mathcal{B}\}$ to $\{\mathcal{I}\}$, parametrized locally by $\boldsymbol{\lambda}$, that is, $\mathcal{R} = \mathcal{R}(\boldsymbol{\lambda})$. Since $\mathcal{R}$ is a rotation matrix, it satisfies the orthonormality condition $\mathcal{R}^T\mathcal{R} = I$. Given the angular velocity vector $\boldsymbol{\omega}$, then $\dot{\boldsymbol{\lambda}} = Q(\boldsymbol{\lambda})\boldsymbol{\omega}$, where $Q(\boldsymbol{\lambda})$ is a matrix that relates the derivative of $\boldsymbol{\lambda}$ with $\boldsymbol{\omega}$. The following kinematic relations apply: $\dot{\mathbf{p}} =^I \mathbf{v} = \mathcal{R}\mathbf{v}$ and $\dot{\mathcal{R}} = \mathcal{R}\mathcal{S}(\boldsymbol{\omega})$, where $\mathcal{S}(\boldsymbol{\omega})$ is a skew symmetric matrix. It is well known that $\mathcal{S}$ satisfies the relationship $\mathcal{S}(a)b = a \times b$, where a, b are arbitrary vectors and x denotes the cross product operation. Furthermore, $||\mathcal{S}(\boldsymbol{\omega})|| = ||\boldsymbol{\omega}||$.

4.2 Time-varying complementary filters. Navigation problem formulation.

We now extend the basic concepts of complementary filtering to the time-varying setting. The motivation for this work can be simply described by considering the example where one is interested in estimating the position $\mathbf{p}$ and velocity ${}^I\mathbf{v}$ of a vehicle based on measurements $\mathbf{p}_m$ and $\mathbf{v}_m$ of $\mathbf{p}$ and $\mathbf{v}$, respectively. In the case of an ocean surface vehicle, $\mathbf{p}_m$ is provided by a Differential Global Positioning System (SGPS), whereas $\mathbf{v}_m$ is provided by a Doppler sonar. In the case of a fully submerged underwater vehicle, $\mathbf{p}_m$ can be provided by a Long Baseline System.

It must be stressed that due to the physical characteristic of the Doppler sonar the *measurement* $\mathbf{v}_m$ *is naturally expressed in body-axis, that is, in the reference frame* $\{\mathcal{B}\}$. Furthermore, Doppler bias effects are also naturally expressed in $\{\mathcal{B}\}$. This is in contrast

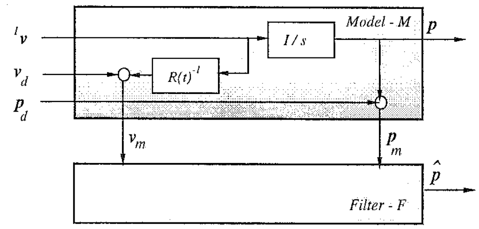

Figure 3: Process model.

with the measurements $\mathbf{p}_m$, which are directly available in the reference frame $\{\mathcal{I}\}$. These facts impose important constraints on the class of complementary filters for position and velocity estimation, as will become clear later. The underlying process model $\mathcal{M}_{pv}$ is depicted in figure 3, where $\mathcal{F}$ is a dynamical system (filter) that operates on the measurements $\mathbf{p}_m$ and $\mathbf{v}_m$ to provide estimates $\hat{\mathbf{p}}$ of $\mathbf{p}$. In the figure, $\mathbf{p}_d$ and $\mathbf{v}_d$ are measurement disturbances. As in the last section, we study the situation where $\mathbf{p}_d = 0$ and $\mathbf{v}_d = \mathbf{v}_{d,0}$ where $\mathbf{v}_{d,0}$ is the Doppler bias. This set-up is all that is required for the design of complementary filters from a "frequency-like" domain point of view. Notice that the process model $\mathcal{M}_{pv}$ is time-varying due to the presence of the rotation matrix $R(t)$. However, the entries of $R(t)$ and their derivatives are not arbitrary functions of time but exhibit bounds that depend on each specific vehicle mission under consideration. We now introduce the following definitions.

Definition. Process Model $\mathcal{M}_{pv}$. The process model $\mathcal{M}_{pv}$ is given by

$$\mathcal{M}_{pv} := \left\{ \begin{array}{rcl} \dot{\mathbf{p}} & = & \mathbf{v} \\ \mathbf{p}_m & = & \mathbf{p} \\ \mathbf{v}_m & = & \mathcal{R}^{-1}\mathbf{v} + \mathbf{v}_{d,0} \end{array} \right. \tag{6}$$

We further assume that the matrix $\mathcal{R}$ and its derivative $\dot{\mathcal{R}}$ are constrained through the inequalities $|\phi(t)| \leq \phi_{max}, |\theta(t)| \leq \theta_{max}, |p(t)| \leq p_{max}, |q(t)| \leq q_{max}$, and $|r(t)| \leq r_{max}$ for all $t \in \mathcal{R}_+$. Notice in the definition above that there are constraints on the roll and pitch angles ϕ and θ respectively, but not on the yaw angle ψ. This is due to the fact ocean vehicles are designed to undergo arbitrary maneuvers in yaw, but pitch and roll excursions are restricted by vehicle construction.

Definition. Candidate complementary filter. Consider the process model $\mathcal{M}_{pv}$ in (6) with $\mathbf{v}_{d,0}$ an arbitrary constant, and let $\mathcal{F}$ be a linear time-varying filter with realization

$$\mathcal{F} := \left\{ \begin{array}{rcl} \dot{\mathbf{x}} & = & A(t)\mathbf{x} + B_p(t)\mathbf{p}_m + B_v(t)\mathbf{v}_m \\ \hat{\mathbf{p}} & = & C(t)\mathbf{x}. \end{array} \right. \tag{7}$$

Then, $\mathcal{F}$ is said to be a *candidate complementary filter* for $\mathcal{M}_{pv}$ if

- $\mathcal{F}$ is internally stable
- For every initial conditions $\mathbf{p}(0)$ and $\mathbf{x}(0)$, $\lim_{t\to\infty}\{p(t) - \hat{p}(t)\} = 0$.
- $\mathcal{F}$ satisfies a bias rejection property, that is, $\lim_{t\to\infty}\hat{\mathbf{p}} = 0$ when $\mathbf{v} = 0$.

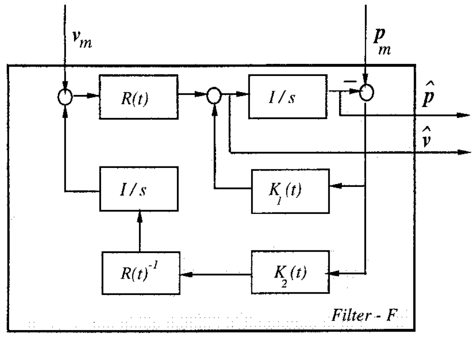

Figure 4: Complementary filter.

Definition. Complementary filter with break frequency ω_c. Let $\mathcal{F}$ be a candidate complementary filter for $\mathcal{M}_{pv}$, and let $\mathcal{F}_p$ denote the corresponding operator from $\mathbf{p}_m$ to $\hat{\mathbf{p}}$. Then, $\mathcal{F}$ is said to be an (ϵ, n) *complementary filter* for $\mathcal{M}_{pv}$ with break frequency ω_c if $\mathcal{F}_p$ is an (ϵ, n) *low pass filter* with bandwidth ω_c.
The discussion in the previous sections leads directly to the following filter design problem.
Problem formulation. *Given the process model* $\mathcal{M}_{pv}$ *in (6) and positive numbers* ω_c, n, *and* ϵ, *find an* (ϵ, n) *complementary filter for* $\mathcal{M}_{pv}$ *with break frequency* ω_c.

5 Complementary filter design. Main results.

This section introduces a specific candidate complementary filter structure for $\mathcal{M}_{pv}$ and derives sufficient conditions for the existence of a complementary filter with the structure adopted that meets required bandwidth constraints.

5.1 Candidate complementary filter structure.

Figure 4 depicts the candidate filter structure for $\mathcal{M}_{pv}$ that will be adopted in the paper. Notice the presence of an extra integrator that was inserted to estimate the rate gyro bias. The filter includes explicitly the rotation matrix $\mathcal{R}(t)$, which we assume is available from an attitude reference system. The following result is obtained.
Theorem 5.1 *Consider the process model* $\mathcal{M}_{pv}$ *and the time-varying filter*

$$\mathcal{F} := \begin{cases} \dot{\mathbf{x}}_1 &= R\mathbf{v}_m + R\mathbf{x}_2 + K_1(\mathbf{p}_m - \mathbf{x}_1) \\ \dot{\mathbf{x}}_2 &= R^{-1}K_2(\mathbf{p}_m - \mathbf{x}_1) \\ \hat{\mathbf{p}} &= \mathbf{x}_1 \end{cases} \tag{8}$$

Suppose the filter $\mathcal{F}$ *is internally stable. Then,* $\mathcal{F}$ *is a candidate complementary filter for* $\mathcal{M}_{pv}$.
Notice that the state $\mathbf{x}_2$ of the appended integrator tends asymptotically to $-\mathbf{v}_{d,0}$. Thus, $\mathbf{x}_2$ provides an estimate of the Doppler bias in the body frame. This result makes perfect sense form a physical point of view since the bias is constant in the body frame (not in the reference frame $\mathcal{I}$).

5.2 The candidate complementary filter: sufficient conditions for stability and guaranteed break frequency.

The next result establishes sufficient conditions for the existence of fixed gains K_1 and K_2 such that the candidate filter is internally stable and has a guaranteed break frequency of at least ω_c, where ω_c is a design parameter. In preparation for that result we let $\boldsymbol{\omega}_r = [p_r \;\; q_r \;\; r_r \;]^T := \mathcal{R}\boldsymbol{\omega}$ and define $\mathcal{S}_r := S(\boldsymbol{\omega}_r) = S(\mathcal{R}\boldsymbol{\omega})$. Given the original design bounds on $\phi(t), \theta(t), p(t), q(t)$, and $r(t)$ it is possible to compute positive upper bounds p_r^+, q_r^+, and r_r^+ such that $|p_r| \leq p_r^+, |q_r| \leq q_r^+, |r_r| \leq r_r^+$. Let $p_r^- = -p_r^+, q_r^- = -q_r^+, r_r^- = -r_r^+$ and construct the set $\{\boldsymbol{\omega}_r^i, i = \{1,..,8\}\}$, where

$$\boldsymbol{\omega}_r^1 = \begin{bmatrix} p_r^- \\ q_r^- \\ r_r^- \end{bmatrix}, \boldsymbol{\omega}_r^2 = \begin{bmatrix} p_r^+ \\ q_r^- \\ r_r^- \end{bmatrix}, ...\boldsymbol{\omega}_r^8 = \begin{bmatrix} p_r^+ \\ q_r^+ \\ r_r^+ \end{bmatrix}.$$

Then $\boldsymbol{\omega}_r \in \mathbf{Co}\{\boldsymbol{\omega}_r^i, i = \{1,..,8\}\}$ and $\mathcal{S}_r \in \mathbf{Co}\{\mathcal{S}_r^i = \mathcal{S}(\boldsymbol{\omega}_r^i);\ i = \{1,..,8\}\}$.
Theorem 5.2 *Consider the linear time-varying filter (8) and assume that the bounds* p_r^+, q_r^+, *and* r_r^+ *on* $\boldsymbol{\omega}_r$ *apply. Given* n *and* ω_c, *let* $\mathcal{W}_{\omega_c}^n := \{A_W, B_W, C_W\}$ *be a minimal realization for the weighting Chebyschev filter introduced in Section 2.2. Further let*

$$F_i = \begin{bmatrix} 0 & I \\ 0 & \mathcal{S}(\boldsymbol{\omega}_r^i) \end{bmatrix}; \quad i = \{1,...8\}, \quad H = [-I \;\; 0].$$

Suppose that given $\epsilon > 0 \;\exists\; M \in \mathcal{R}^{6\times 3}$, $P_1 \in \mathcal{R}^{6\times 6}, P_2 \in \mathcal{R}^{6\times 6}$, $P_1 > 0$, $P_2 > 0$ *such that the* linear matrix inequalities

$$\begin{bmatrix} Q_1 & Q_2 & 0 \\ Q_2^T & Q_3 & Q_4 \\ 0 & Q_4^T & -\epsilon^2 I \end{bmatrix} < 0,$$

are satisfied, where $Q_1 = F_i^T P_1 + H^T M^T + P_1 F_i + MH + H^T H$, $Q_2 = MC_W + H^T C_W$, $Q_3 = P_2 A + A^T P_2 + C_W^T C_W$, *and* $Q_4 = P_2 B_W$. *Then, the constant gains*

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = P_1^{-1} M$$

make the filter $\mathcal{F}$ *internally stable. Furthermore, the operator* $\mathcal{F}_p : \mathbf{p} \to \hat{\mathbf{p}}$ *satisfies a low pass property with indices* (ϵ, n) *over* $[0, \omega_c]$, *that is,* $||(\mathcal{F}_p - I)\, W_{\omega_c}^n|| < \epsilon$.
The reader will find in [12] the details of converting the above analysis result into a synthesis method that builds on numerical algorithms that are available with the LMI Toolbox for Matlab [9].

6 Filter Design: a practical algorithm. Simulation results.

The previous section introduced the mathematical tools that are required to design a candidate complementary filter with a guaranteed break frequency. Notice, however, that the outcome of the design process may very well be a filter with an effective bandwidth that is greater than the one required. Clearly, the set

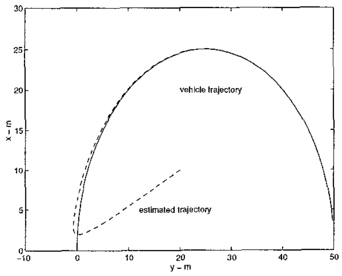

Figure 5: Actual and estimated vehicle trajectory.

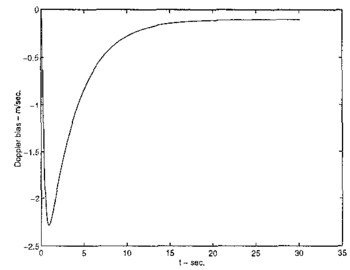

Figure 6: - Doppler bias estimate.

of possible solutions must be further constrained so that the designer have an extra design parameter at his disposal to select one solution (if it exists) that meets the required break frequency criterion. This situation is identical to what happens in the case of filter design using Kalman-Bucy theory, where the noise covariances play the role of "tuning knobs" to shape the filter characteristics.

In the linear time-invariant case, a simple analysis of a Bode diagram indicates that an expedite way of setting an upper bound on the break frequency is to make the filter "roll-off" sufficiently early in the frequency. In the time-varying setting, this corresponds to making $||\mathcal{F}_p(I - \mathcal{W}_{\omega_t}^{n_t})||$ sufficiently small for adequate choices of ω_t and n_t, which play the role of "tuning parameters". These considerations lead directly to a practical algorithm for complementary filter design whereby theorem 5.2 is used with the additional constraint introduced above, which can be easily cast as a Linear Matrix Inequality. It is then up to the system designer to select appropriate values of the tuning parameters to try and meet all the criteria that are required for a complementary filter.

To illustrate the performance of the new complementary filtering structure, a simple filter design exercise was carried out for an autonomous surface vehicle undergoing rotational maneuvers in the horizontal plane. In this case, the navigation system is required to provide accurate estimates of the vehicle's position based on position and velocity measurements provided by a DGPS and a Doppler sonar, respectively. In the scenario adopted the vehicle progresses at a constant speed of $2m/s$ while it executes repeated turns at a maximum yaw rate of $3rad/s$. The Doppler sonar is assumed to introduce a constant bias term $\mathbf{v}_{d,0} = [0.1m/s,\ 0.2m/s]^T$. The selected break frequency for the complementary filter was $\omega_c = 0.1rad/s$. Figure 5 shows the actual and estimated vehicle position when the initial state of the filter was set to $\mathbf{x}_1 = [10m,\ 20m]^T$ and $\mathbf{x}_2 = [0m/s,\ 0m/s]^T$. Figure 6 captures the evolution of the first component of the Doppler bias estimate. It can be concluded from the figures that the filter provides good tracking of the actual inertial trajectory and rejects the bias introduced by the Doppler unit in the body-axis.

References

[1] I. Bar-Itzhack and I. Ziv, "Frequency and time domain designs of strapdown vertical determination systems," *Proc. AIAA Guidance, Navigation and Control*, Williamsburgh, August 1986, pp. 505-515.

[2] R. Brown and P. Hwang. *Introduction to Random Signals and Applied Kalman Filtering.* Second Edition, John Wiley and Sons, Inc., 1992.

[3] R. Brown, "Integrated navigation systems and Kalman filtering: a perspective," *Journal of the Institute of Navigation*, Winter 1972-73, Vol. 19, N0.4, pp.355-362.

[4] S. Boyd, L. El Ghaoui, E. Feron and B. Balakrishnan. *Linear Matrix Inequalities in Systems and Control Theory.* SIAM Studies in Applied Mathematics, Philadelphia, 1994.

[5] Fryxell, D., P. Oliveira, A. Pascoal, C. Silvestre, and I. Kaminer, "Navigation, guidance, and control systems of AUVS: an application to the MARIUS vehicle," *IFAC Control Engineering Practice*, pp. 401-409, March 1996.

[6] G. Goodwin and M. Seron, "Fundamental design tradeoffs in filtering, prediction, and smoothing," *IEEE Trans. Automatic Control*, Vol. 42, N0.9, September 1997, pp.1240-1251.

[7] K. Grigoriadis and J. Watson, "Reduced-order H_∞ and $L_2 - L_\infty$ filtering via linear matrix inequalities," *IEEE Trans. Aerospace and Electronic Systems*, Vol. 33, N0.4, October 1997, pp. 1326-1338.

[8] D. Jourdan, "Doppler sonar navigation error propagation and correction" *Journal of the Institute of Navigation*, Vol. 32,No.1, Spring 1985, pp.29-56.

[9] MatLab Application Toolbox, LMI Control. *The Math Works Inc.*,1997.

[10] S. Merhav. *Aerospace Sensor Systems and Applications.* Springer-Verlag, 1996.

[11] K. Nagpal and P. Khargonekar, "Filtering and smoothing in an H_∞ setting," *IEEE Trans. Automatic Control*, Vol. 36, 1991, pp.152-166.

[12] A. Pascoal, I. Kaminer, and P. Oliveira, "On the design of Time-Varying Complementary Filters for Autonomous Vehicle Navigation," Joint Internal Report IST/NPS, Feb. 1997.

[13] G. Siouris. *Aerospace Avionics Systems: a Modern Synthesis.* Academic Press Inc., 1993.

[14] J. Stambaugh and R. Thibault, "Navigation requirements for autonomous underwater vehicles," *Journal of the Institute of Navigation*, Vol. 39, No.1, Spring 1992, pp.79-92.

[15] M. Vidyasagar. *Nonlinear System Analysis.* Prentice-Hall, Inc. 1978.