Globally exponentially stable filters for source localization and navigation aided by direction measurements

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A B S T R A C T

This paper presents a set of filters with globally exponentially stable error dynamics for source localization and navigation, in 3-D, based on direction measurements from the agent (or vehicle) to the source, in addition to relative velocity readings of the agent. Both the source and the agent are allowed to have constant unknown drift velocities and the relative drift velocity is also explicitly estimated. The observability of the system is studied and realistic simulation results are presented, in the presence of measurement noise, that illustrate the performance of the achieved solutions. Comparison results with the Extended Kalman Filter are also provided and similar performances are achieved.

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1. Introduction

The problem of source localization has been the subject of intensive research in recent years. Roughly speaking, an agent has access to a set of measurements and aims to estimate the position of a source. The set of measurements depends on the environment in which the operation occurs and the mission scenario itself. Previous work in the field can be found in [1], where the authors propose a localization algorithm based on the range to the source and the inertial position of the agent. Global exponential stability (GES) is achieved under a persistent excitation condition and the analysis is extended to the case of a non-stationary source, where it is shown that it is possible to achieve tracking up to some bounded error. In [2] the same problem was addressed considering, in addition to range readings to the source, relative velocity readings of the agent. The observability of the system was assessed, including also relative drift velocities, and filtering solutions were proposed with globally asymptotically stable error dynamics. More recently, in [3], the same problem was addressed, in 2-D, based on bearing measurements, in addition to the trajectory of the agent. The estimation error dynamics were shown to be GES under an appropriate persistent excitation condition and a circulating control law was also proposed. Earlier work on the observability issues of target motion analysis based on angle readings, in 2-D, can be found in [4], which was later extended to 3-D in [5]. The specific observability criteria thereby derived resort to complicated nonlinear differential equations and some tedious mathematics are needed for the solution, giving conditions that are necessary for system observability. Another related framework in the domain of target motion analysis (TMA) can be found in [6], where frequency measurements are also included. This topic was further studied in [7], where Cramer–Rao analysis revealed the parametric dependencies of TMA with angle-only tracking and angle/frequency tracking, giving also an idea of the increase in estimation accuracy using the later.

Parallel to the topic of source localization based on range or bearing measurements is the topic of navigation aided by these sensors. Previous work by the authors with range measurements can be found in [8], where acceleration readings were also considered. The observability of the system was assessed and conditions were derived that guarantee globally asymptotically stable error dynamics. In [9] a similar design was proposed with two vehicles working in tandem considering relative velocity drifts. Globally asymptotically stable error dynamics were also shown under appropriate observability conditions. In [10] the authors deal with the problem of underwater navigation in the presence of unknown
currents based on range measurements to a single beacon. An
observer analysis is presented based on the linearization of
the nonlinear system which yields local results. Based on the
linearized system dynamics, a Luenberger observer is introduced but
in practice an Extended Kalman filter (EKF) is implemented, with
no warranties of global asymptotic stability. More recently, the
same problem has been studied in [11,12], where EKFs have been
extensively used to solve the navigation problem based on single
beacon range measurements. The problem of localization of a mo-
tile robot using bearing measurements was also addressed in [13],
where a nonlinear transformation of the measurement equation
into a higher dimensional space is performed. This has allowed to
obtain tight, possibly complex-shaped, bounding sets for the feasi-
bility states in a closed-form representation.

This paper addresses the problem of navigation/source local-
ization based on direction measurements to a single source in the
presence of unknown constant drifts. The observability of the sys-
tem is studied and Kalman filters with EKF error dynamics are pro-
posed, without system linearizations and yielding performances
comparable to those of the Extended Kalman Filter but with GE
guarantees. Central to the design is the augmentation of the system
central to the knowledge of the source velocity.

In addition to more detailed explanations and further discussion
of issues such as the linear meaning of the observability condi-
tions, the present paper acknowledges GE error dynamics and in-
cludes an additional solution for navigation that does not require
the knowledge of the source velocity.

1.1. Notation

Throughout the paper the symbol 0 denotes a matrix (or vec-
tor) of zeros and I an identity matrix, both of appropriate dimen-
sions. A block diagonal matrix is represented as diag(A1, . . . , Am)
and the set of unit vectors on R3 is denoted by S(2). Finally, δ(t)
corresponds to the Dirac delta function.

2. Problem statement

2.1. Source localization

Let \( p(t) \in R^3 \) denote the position of a point-mass agent, in
ertial coordinates, moving in a scenario where there is a source
whose position, in inertial coordinates, is denoted by \( s(t) \in R^3 \).
Suppose that the source is moving with constant unknown velocity
\( v_s(t) \in R^3 \) relative to the inertial frame, which gives
\( s(t) = v_s(t) + s_0 \) and \( \dot{v}_s(t) = 0 \), while the linear motion
kinematics of the agent are given by \( \dot{p}(t) = v_s(t) + \dot{v}_s(t) \) and \( \dot{v}_s(t) = 0 \), where \( v_s(t) \in R^3 \) is a constant
unknown drift velocity of the agent and \( v_s(t) \in R^3 \) is a known
central to the position. In the context of the EU project TRIDENT, the
source is an Autonomous Surface Craft (ASC) and the agent an Au-
onomous Underwater Vehicle (AUV). The ASC is moving with con-
stant unknown velocity \( v_s(t) \) and the AUV is moving with velocity
relative to the source \( v_A(t) \), as given by a Doppler Velocity Log
(DVL), in the presence of constant unknown ocean currents with velocity
\( v_c(t) \). Further consider that the agent measures the direction to the
source

\[
\mathbf{d}(t) = \frac{\mathbf{r}(t)}{\| \mathbf{r}(t) \|} \in S(2),
\]

with \( \mathbf{r}(t) := s(t) - p(t) \in R^3 \). The problem of source local-
ization considered here is that of estimating the position of the
source relative to the agent, \( \mathbf{r}(t) \), and the relative drift velocity
\( \mathbf{v}_d(t) := \mathbf{v}_s(t) - \dot{v}_s(t) \in R^3 \), given direction and relative velocity
readings, \( \mathbf{d}(t) \) and \( \dot{v}_s(t) \), respectively. The corresponding system
dynamics are given by

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
\dot{y}(t) &= C(t)x(t),
\end{align*}
\]

where

\[
A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in R^{7 \times 7},
\]

\[
B(t) = \begin{bmatrix} -1 \\ 0 \\ -d(t) \end{bmatrix} \in R^{7 \times 3},
\]

\[
C(t) = \begin{bmatrix} 1 & 0 & -d(t) \end{bmatrix} \in R^{3 \times 7},
\]

\( u(t) = v_s(t). \)

2.2. Navigation

In the context of the EU project TRIDENT, an ASC and an AUV
work in close cooperation in order to achieve a certain goal. Assume
that the ASC (the source) transmits its inertial position \( s(t) \) and
velocity \( v_s(t) \) to the AUV (the agent). In this framework, the
goal of the AUV (the agent) is now to determine its own position
in inertial coordinates \( p(t) \), as well as its drift velocity \( \dot{v}_s(t) \), given
the information provided by the ASC (the source), the relative ve-
locity readings \( v_s(t) \), and the direction measurements \( \mathbf{d}(t) \). In this
framework \( v_s(t) \) is no longer required to be constant and the sys-
tem dynamics are given by

\[
\begin{align*}
\dot{p}(t) &= v_s(t) + \dot{v}_s(t) \\
\dot{v}_s(t) &= 0 \\
\dot{d}(t) &= \| s(t) - p(t) \|
\end{align*}
\]

3. Source localization filter design

3.1. System dynamics

In order to derive an augmented linear time-varying system for source
localization, consider the system states \( x_1(t) := r(t), x_2(t) := v_s(t), \) and
\( x_3(t) := \| r(t) \| \) and define the state vector
\( x(t) := \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}^T \in R^7 \). From (1) it follows that \( x_1(t) =
\begin{align*}
x(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t),
\end{align*}
\]

where

\[
A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \end{bmatrix} \in R^{7 \times 7},
\]

\[
B(t) = \begin{bmatrix} -1 \\ 0 \\ -d(t) \end{bmatrix} \in R^{7 \times 3},
\]

\[
C(t) = \begin{bmatrix} 1 & 0 & -d(t) \end{bmatrix} \in R^{3 \times 7},
\]

3.2. Observability analysis

The observability of the problem of source localization with
relative velocity readings and direction measurements is studied
in this section. The following proposition (Proposition 4.2, [15]) is useful in the sequel.

Proposition 1. Let \( f(t) : \mathbb{R} \to \mathbb{R}^n \) be a continuous and i-times continuously differentiable function on \( I := [t_0, t_f] \), \( i := t_0 - t_f > 0 \), and such that \( f(t_0) = \hat{f}(t_0) = \cdots = f^{(i-1)}(t_0) = 0 \). Further assume that there exists a nonnegative constant \( c \) such that \( |f^{(i)}(t)| \leq c \) for all \( t \in I \). If there exist \( \alpha > 0 \) and \( t_1 \in I \) such that \( |f^{(i-1)}(t_1)| \geq \alpha \), then there exist \( 0 < \delta \leq T \) and \( \beta > 0 \) such that \( |f(t_0 + \delta)| \geq \beta \).

The following theorem characterizes the observability of the LTV system (3).

Theorem 1. The LTV system (3) is observable on \( I := [t_0, t_f] \) if and only if the unit vector \( d(t) \) is not constant on \( I \) or, equivalently,

\[
\exists_{t \in I} \quad d^T(t_0) d(t_1) < 1.
\]

Proof. The observability Gramian associated with the pair \((A(t), C(t))\) on \( I \) is given by

\[
W(t_0, t_f) = \int_{t_0}^{t_f} \phi(t, \sigma) C(t) dA(t) d\tau,
\]

where \( \phi(t, t_0) \) is the transition matrix associated with \( A(t) \).

\[
\phi(t, t_0) = \begin{bmatrix} 1 & (t - t_0) & 0 \\ 0 & 1 & 0 \\ 0 & \int_{t_0}^{t} d^T(\tau) d\tau & 1 \end{bmatrix} \in \mathbb{R}^{2 \times 2}.
\]

Let \( c = [c_1, c_2, c_3]^T \in \mathbb{R}^2, c_i \in \mathbb{R}, i = 1, 2, c_1 \in \mathbb{R}, \) be a unit vector, i.e., \( ||c|| = 1 \). Then,

\[
c^T W(t_0, t_f) c = \int_{t_0}^{t_f} ||f(\tau)||^2 d\tau
\]

for all \( ||c|| = 1 \), where

\[
f(\tau) = c_1 + \left( (t - t_0) I - d(t) \int_{t_0}^{t} d^T(\sigma) d\sigma \right) c_2 - c_3 d(t)
\]

for all \( t \in I \). The first two derivatives of \( f(\tau) \) are given by

\[
\frac{df(\tau)}{d\tau} = \left[ 1 - d(t) d^T(\tau) - d(\tau) \int_{t_0}^{t} d^T(\sigma) d\sigma \right] c_2 - c_3 d(t)
\]

and

\[
\frac{d^2f(\tau)}{d\tau^2} = \left[ -2d(t) d^T(\tau) - d(\tau) d^T(\tau) \right] c_2 - c_3 d(t)
\]

for all \( \tau \in I \). Notice that, under Assumption 1, both derivatives are norm-bounded, from above, on \( I \).

The proof of necessity follows by contraposition. Suppose that (4) is not verified. Then, the unit vector \( d(t) \) is constant on \( I \), i.e., \( d(t) = d(t_0) \) for all \( t \in I \). Let \( c_1 = \frac{d(t_0)}{d(t_0)}, c_2 = 0, \) and \( c_3 = \frac{d(t)}{d(t_0)} \). Then, it follows that \( f(\tau) = \frac{d(t_0)}{d(t_0)} d(t_0) - \frac{d(t_0)}{d(t_0)} d(t_0) = 0 \) for all \( \tau \in I \), which in turn allows to conclude that the observability Gramian \( W(t_0, t_f) \) is not invertible and the LTV system (3) is not observable on \( I \). Consequently, if the LTV system (3) is observable on \( I \), it follows that (4) is true.

To show that (4) is also a sufficient condition, suppose first that \( c_3 \neq 0 \). Then, if \( c_1 = c_3 d(t_0) \), it follows that \( ||f(t_0)|| > 0 \) and, from Proposition 1, it must be \( c^T W(t_0, t_1) c > 0 \). Consider now \( c_1 = c_3 d(t_0), \) with \( c_3 \neq 0 \). In this case, \( f(t_0) = 0 \) and

\[
\frac{df(t_0)}{d\tau} = \left[ 1 - d(t_0) d^T(t_0) \right] c_2 - c_3 d(t_0).
\]

If \( \int_{t_0}^{t_f} f(t_0) d\tau > 0 \), it follows, using Proposition 1 twice, that \( c^T W(t_0, t_f) c > 0 \). Otherwise, if \( \frac{df(t_0)}{d\tau} = 0 \), two cases may be considered: (i) if \( d(t_0) = 0 \), it may be \( c_2 = 0 \) or \( c_2 = c_3 d(t_0) \) for some scalar \( c_2 \); or (ii) if \( d(t_0) \neq 0 \), it must be \( c_2 = c_3 d(t_0) \), where it is used the fact that \( d^T(t_0) d(t_0) = 0 \) for all \( t \). Evaluating \( f(t_0) \) at \( t = t_1 \), when \( c_2 = 0 \), yields \( f(t_1) = c_3 d(t_0) - c_3 d(t_1) \) which has a positive norm if (4) is true. As such, it follows from Proposition 1 that \( c^T W(t_0, t_f) c > 0 \) for \( c_1 = c_3 d(t_0) \), \( c_2 = 0 \), \( c_3 \neq 0 \).

If \( c_2 = c_3 d(t_0), f(t_1) \) reads as

\[
f(t_1) = \left[ c_1 + c_3 (t_1 - t_0) \right] d(t_0) - \left[ c_1 + c_2 \int_{t_0}^{t_1} d^T(\sigma) d(t_0) d\sigma \right] d(t_1),
\]

where \( d(t_0 + \epsilon) = c_2 d(t_0) + c_3 d(t_0) \), and, using Proposition 1, it must be \( c^T W(t_0, t_f) c > 0 \) for \( c_1 = 0, c_2 = \pm d(t_0), c_3 = 0 \). This allows to conclude, so far, that if \( c_3 \neq 0 \), \( c^T W(t_0, t_f) c > 0 \). It remains to see what happens when \( c_3 = 0 \). If \( c_1 \neq 0 \), it turns out that \( ||f(t_0)|| > 0 \) and again, using Proposition 1, it must be \( c^T W(t_0, t_f) c > 0 \) for \( c_1 \neq 0, c_2 = 0 \). On the other hand, if \( c_1 = 0, c_3 = 0 \), it follows that \( f(t_0) = 0 \) and

\[
\frac{df(t_0)}{d\tau} = \left[ 1 - d(t_0) d^T(t_0) \right] c_2.
\]

Now, if \( c_2 \neq \pm d(t_0) \), it follows that

\[
\left| \frac{df(t_0)}{d\tau} \right| = \left[ 1 - d(t_0) d^T(t_0) \right] c_2 > 0
\]

and, using Proposition 1 twice, it must be \( c^T W(t_0, t_f) c > 0 \) for \( c_1 = 0, c_2 = \pm d(t_0), c_3 = 0 \). But this concludes the proof, as it is shown that \( c^T W(t_0, t_f) c > 0 \) for all \( ||c|| = 1 \), which means that the observability Gramian in invertible and as such (3) is observable.

Before proceeding, it is important to remark that there is nothing in (3) imposing the nonlinear restriction \( ||x(t)|| = x_1(t) = ||r(t)|| \). This is true, by construction, if it is satisfied for \( t = t_0 \). The following theorem addresses this issue.
Theorem 2. Under the hypothesis of Theorem 1, the initial condition of the LTV (3) corresponds to the initial condition of the original nonlinear system, i.e.,

\[
\begin{aligned}
\{ x_1 (t_0) = r (t_0) \\
x_2 (t_0) = v_m (t_0) \\
x_3 (t_0) = \| r (t_0) \|.
\end{aligned}
\]

(5)

Proof. Under the terms of Theorem 1, the initial condition of the LTV system (3) is uniquely determined by the corresponding system output and input. The proof follows by showing that (5) explains the system output. As the initial condition is uniquely determined, if (5) explains the output of the system, it must correspond to the initial condition. The output of the LTV system (3) is given by

\[
y (t) = x_1 (t_0) + (t - t_0) x_2 (t_0) - \int_{t_0}^{t} u (\tau) d \tau - x_3 (t_0) d (t)
\]

\[
- \int_{t_0}^{t} [x_2 (t_0) - u (\tau)]^T d (\tau) d \tau d (t) = 0
\]

(6)

for all \( t \in I \), \( I = [t_0, T] \). Substituting (5) in (6) gives

\[
y (t) = r (t_0) - \| r (t_0) \| d (t) + \int_{t_0}^{t} [v_m (t_0) - u (t)] d \tau
\]

\[
- \int_{t_0}^{t} [v_m (t_0) - u (t)]^T d (\tau) d \tau d (t).
\]

(7)

As \( v_m (t) \) is constant, it is possible to rewrite (8) as

\[
y (t) = - \left[ \| r (t_0) \| + \int_{t_0}^{t} [v_m (t_0) - u (t)]^T d (\tau) d \tau \right] d (t)
\]

\[
+ [v_m (t_0) - u (t)] - [v_m (t_0) - u (t)]^T d (t) d (t).
\]

(8)

As \( v_m (t) \) is constant, it is possible to rewrite (8) as

\[
y (t) = - \left[ \| r (t_0) \| + \int_{t_0}^{t} [v_m (t_0) - u (t)]^T d (\tau) d \tau \right] d (t)
\]

\[
+ [v_m (t_0) - u (t)] - [v_m (t_0) - u (t)]^T d (t) d (t).
\]

(9)

Using the derivative \( \frac{d}{dt} \| r (t) \| = \| v_m (t) - u (t) \|^T d (t) \), allows to write

\[
\| r (t) \| = \| r (t_0) \| + \int_{t_0}^{t} [v_m (t) - u (t)]^T d (\tau) d \tau.
\]

(10)

On the other hand, the time derivative of (1) is given by

\[
d (t) = \frac{[v_m (t) - u (t)] - [v_m (t_0) - u (t)]^T d (t) d (t)}{\| r (t) \|}.
\]

(11)

Substituting (10) and (11) in (9) gives \( y (t) = 0 \). This concludes the proof, as with \( y (t_0) = 0 \) and \( y (t) = 0 \) it must be \( y (t) = 0 \) for all \( t \in I \) and therefore (5) is true. \( \Box \)

In order to design GES observers (or filtering) solutions, stronger forms of observability are convenient. The following theorem addresses this issue.

Theorem 3. The LTV system (3) is uniformly completely observable if and only if

\[
\exists \alpha > 0, \forall t \geq t_0 \int_{t}^{t + \delta} d (t) d (t) d \tau \leq \delta (1 - \alpha).
\]

(12)

Proof. The proof of sufficiency follows similar steps to Theorem 1 considering uniformity bounds that stem from the persistent excitation condition (12). Therefore it is omitted. To show that (12) is also necessary, suppose that (12) does not hold. Then,

\[
\forall \alpha > 0, \exists t^* \geq t_0 \int_{t}^{t + \delta} d (t) d (t) d \tau \geq \delta (1 - \alpha).
\]

(13)

Let \( c = \left[ \frac{\sqrt{2}}{2} d (t^*) \quad 0 \quad \sqrt{2}/2 \right]^T \in \mathbb{R}^7 \). Then,

\[
c^T W (t^*, t^* + \delta) c = \frac{1}{2} \int_{t^*}^{t^* + \delta} \| d (t^*) - d (t) \|^2 d \tau
\]

\[
= \frac{1}{2} \int_{t^*}^{t^* + \delta} \left[ \| d (t^*) \|^2 + \| d (t) \|^2 - 2 d (t^*) d (t) \right] d \tau.
\]

(14)

As \( d (t) \) is a unit vector, it is possible to write (14) as

\[
c^T W (t^*, t^* + \delta) c = \delta - \int_{t^*}^{t^* + \delta} d (t^*) d (t) d \tau.
\]

(15)

Using (13) in (15) allows to conclude that for all \( \alpha > 0 \) and \( \delta > 0 \) there exists time instant \( t^* \geq t_0 \) such that \( c^T W (t^*, t^* + \delta) c < \delta \alpha \), which means that the LTV system (3) is not uniformly completely observable. Therefore, if the LTV system (3) is uniformly completely observable, (12) is true. \( \Box \)

For observability over a fixed time interval Theorem 1 already provides sufficient insight: the system is observable if the direction measurements do not remain constant on that interval. For uniform complete observability the result provided by Theorem 3 is essentially an extension considering uniformity in time: the system is uniformly completely observable if it is possible to choose a fixed time interval length \( \delta > 0 \) such that, for all time intervals of length \( \delta \), there is a minimum variation in the direction measurements, uniformly in time, which is encoded by the positive constant \( \alpha \) in (12).

3.3. Kalman filter

Section 3.1 introduced a LTV system for source localization and its observability was characterized in Section 3.2. In particular, it was shown that the LTV system (3) is uniformly completely observable if and only if an appropriate persistent excitation condition, (12), is satisfied. As such, the design of a Kalman filter, with globally exponentially stable error dynamics, follows naturally. An alternative observer with globally exponentially stable error dynamics could be devised using [16, Theorem 15.2]. Considering additive system disturbances and sensor noise, the system dynamics are given by

\[
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + w(t) \\
\dot{y}(t) &= C(t)x(t) + n(t),
\end{aligned}
\]

where \( w(t) \in \mathbb{R}^7 \) is zero-mean white Gaussian noise, with \( E [w(t) w(t)^T (t - \tau)] = \Xi \delta (\tau), \Xi > 0 \), \( n(t) \in \mathbb{R}^3 \) is zero-mean white Gaussian noise, with \( E [n(t) n(t)^T (t - \tau)] = \Theta \delta (\tau), \Theta > 0 \), and \( E [w(t) n(t)^T (t - \tau)] = 0 \). It is important to stress, however, that it is not possible to conclude that this is an optimal solution, as the actual system disturbances and sensor noise may not be additive. Nevertheless, the nominal filter error dynamics are globally exponentially stable if the LTV system is uniformly completely observable and controllable [17]. The design of the Kalman filter is well known and therefore it is omitted.

Remark 1. Even though the drift velocities are assumed, in nominal terms, as constant, it is possible to track slowly time-varying
drift velocities (up to some error) by appropriate tuning of the corresponding state disturbance covariance design parameter of the Kalman filter.

4. Navigation filter design assuming known source velocity

This section presents a solution for navigation based on direction measurements similar to the solution for source localization proposed in Section 3. In order to derive an augmented linear time-varying system for navigation based on direction readings, define the system states \( x_1(t) = p(t), x_2(t) = v(t), \) and \( x_3(t) = \|r(t)\| \). From (1) it follows that \( x_1(t) + x_2(t) d(t) = s(t) \) for all \( t \). Let \( x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) x_3(t) \end{pmatrix} \in \mathbb{R}^7 \). Then, the system dynamics are given by the LTV system

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t),
\end{align*}
\]

where

\[
A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -d^T(t) & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 7},
\]

\[
B(t) = \begin{bmatrix} 1 \\ 0 \\ -d^T(t) & d^T(t) \end{bmatrix} \in \mathbb{R}^{3 \times 6},
\]

\[
C(t) = \begin{bmatrix} 1 & 0 & d(t) \end{bmatrix} \in \mathbb{R}^{3 \times 1}, \quad u(t) = \begin{bmatrix} v^T(t) \\ v^T(t) \end{bmatrix} \in \mathbb{R}^6.
\]

Next, consider the Lyapunov state transformation of the LTV system (16) given by \( z(t) = \text{diag}(I, I, -1)x(t) \), which preserves observability properties. The new system dynamics read as

\[
\begin{align*}
\dot{z}(t) &= A(t)z(t) + \text{diag}(I, I, -1)B(t)u(t) \\
y(t) &= C(t)z(t).
\end{align*}
\]

Notice that the new system matrices \( A(t) \) and \( C(t) \) are those of the LTV system (3). This immediately allows to characterize the observability of the LTV system (16) with the following two theorems, as both systems are related by a Lyapunov state transformation [18].

**Theorem 4.** The LTV system (16) is observable on \( I := [t_0, t_f] \) if and only if the unit vector \( d(t) \) is not constant on \( I \) or, equivalently, (4) is true.

**Theorem 5.** The LTV system (16) is uniformly completely observable if and only if (12) holds.

It remains to see that, as in the solution for source localization, the initial condition of the LTV system, uniquely determined under the observability condition expressed in the previous theorems, matches the initial condition of the original system. This is shown in the following theorem.

**Theorem 6.** Under the hypothesis of Theorem 4, the initial condition of the LTV (16) corresponds to the initial condition of the original nonlinear system, i.e.,

\[
\begin{align*}
x_1(t_0) &= p(t_0) \\
x_2(t_0) &= v(t_0) \\
x_3(t_0) &= \|r(t_0)\|.
\end{align*}
\]

**Proof.** Under the terms of Theorem 4, the initial condition of the LTV system (16) is uniquely determined by the corresponding system output and input. The proof follows by showing that (17) explains the system output. The output of the LTV system (16) is given by

\[
y(t) = x_1(t_0) + (t - t_0)x_2(t_0) + \int_{t_0}^t v_1(\tau)\,d\tau + x_3(t_0)\,d(t)
\]

\[
+ \int_{t_0}^t [v_1(\tau) - v_1(t)]^T d(\tau)\,d\tau = s(t)
\]

for all \( t \in I \). Substituting (17) in (18) gives

\[
y(t) = p(t_0) + (t - t_0)v(t_0) + \int_{t_0}^t v_1(\tau)\,d\tau + \|r(t_0)\|\,d(t)
\]

\[
+ \int_{t_0}^t [v_1(\tau) - v_1(t)]^T d(\tau)\,d\tau.
\]

It remains only to show that (19) is equal to \( s(t) \) for all \( t \in I \). Substituting \( t = t_0 \) in (19) yields

\[
y(t_0) = p(t_0) + \|r(t_0)\|\,d(t_0) = p(t_0) + r(t_0) = s(t_0).
\]

The time derivative of (19) is given by

\[
y'(t) = \|r(t_0)\|\,d'(t) + v_1(t) + v_1(t)
\]

\[
+ \int_{t_0}^t [v_1(\tau) - v_1(t)]^T d(\tau)\,d\tau + [v_1(t) - v_1(t) - v_1(t)]^T d(\tau)\,d\tau.
\]

As \( v_1(t) \) is constant, it is possible to rewrite (20) as

\[
y'(t) = \|r(t_0)\|\,d'(t) + v_1(t) + v_1(t)
\]

\[
+ \int_{t_0}^t [v_1(\tau) - v_1(t) - v_1(t)]^T d(\tau)\,d\tau + [v_1(t) - v_1(t) - v_1(t)]^T d(\tau)\,d\tau.
\]

Using \( \frac{d}{dt} \|r(t)\| = |v_1(t) - v_1(t) - v_1(t)|^T \) one may write

\[
\|r(t)\| = \|r(t_0)\| + \int_{t_0}^t [v_1(\tau) - v_1(t) - v_1(t)]^T d(\tau)\,d\tau.
\]

On the other hand, the time derivative of (1) is given by

\[
\dot{d}(t) = \frac{v_1(t) - v_1(t) - v_1(t)}{\|r(t)\|} \frac{d(\tau)}{d\tau} - \frac{[v_1(t) - v_1(t) - v_1(t)]^T d(\tau)}{\|r(t)\|^2} \, d(\tau).
\]

Substituting (22) and (23) in (21) gives \( \dot{y}(t) = y(t) \). This concludes the proof, as with \( y(t_0) = s(t_0) \) and \( y(t) = s(t) = s(t) \) it must be \( y(t) = s(t) \) for all \( t \in I \) and therefore (17) is true.

The design of a Kalman filter with globally exponentially stable error dynamics for navigation based on direction measurements follows naturally as in Section 3.3.

5. Navigation filter design without the source velocity

The design for navigation aided by direction measurements presented in Section 4 requires the velocity of the source. Although that is feasible in cooperative navigation, it is also interesting to consider a scenario where \( v_1(t) \) is not available. This could be interesting when the source is equipped with a localization sensor but not a full navigation system. This section deals with this problem, presenting an alternative design for navigation based on direction measurements that does not require the velocity of the source.
To that purpose, notice that
\[ [I - d(t)dd^T(t)] r(t) = [I - d(t)dd^T(t)] v(t) \]
for all \( t \), which allows to write
\[ [I - d(t)dd^T(t)] p(t) = [I - d(t)dd^T(t)] s(t). \]
(24)

Combining (24) with (2) gives the LTV system
\[
\begin{align*}
\dot{x}_f(t) &= A_f x_f(t) + B_f u_f(t) \\
\dot{y}(t) &= C_f x_f(t)
\end{align*}
\]
(25)
where \( x_f(t) = \begin{bmatrix} p(t) & v^T(t) \end{bmatrix}^T \in \mathbb{R}^6 \) is the system state, \( u_f(t) = v(t) \) is the system input,
\[
A_f = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{6 \times 6}, \quad B_f = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{R}^{6 \times 3}
\]
and \( C_f(t) = [I - d(t)dd^T(t) - 0] \in \mathbb{R}^{3 \times 6} \).

It is now important to assess about the observability of the LTV system (25) in order to apply a Kalman filter. It turns out that the results are identical to those previously derived, as detailed in the following theorems.

**Theorem 7.** The LTV system (25) is observable on \( \tau := [t_0, \tau] \) if and only if the unit vector \( d(t) \) is not constant on \( \tau \) or, equivalently, (4) is true.

**Proof.** Let \( c = \begin{bmatrix} c_1^T & c_2^T \end{bmatrix}^T \in \mathbb{R}^6 \), \( c_i \in \mathbb{R}^3, i = 1, 2 \), be a unit vector, i.e., \( \|c\| = 1 \). Then, it is straightforward to show that
\[
\begin{align*}
\frac{d}{d\tau} f(\tau) &= \frac{d}{d\tau} \begin{bmatrix} c_1(t) + (\tau - t_0)c_2(t) \\ c_1(t) + (\tau - t_0)c_2(t) \end{bmatrix} \\
&= \begin{bmatrix} c_1(t) + (\tau - t_0)c_2(t) \\ c_1(t) + (\tau - t_0)c_2(t) \end{bmatrix}
\end{align*}
\]
for all \( \tau \). For the proof of necessity follows by contraposition. Suppose that (4) is not verified. Then, the unit vector \( d(t) \) is constant on \( \tau \), i.e., \( d(t) = d(t_0) \) for all \( t \). Let \( c = d(t_0), c_2 = 0 \). Then, \( f(\tau) = \begin{bmatrix} 1 - d(t_0)d^T(t_0) \\ 0 \end{bmatrix} \) for all \( \tau \), which in turn allows to conclude that the observability Gramian \( W_0(t_0, t_1) \) is not invertible and the LTV system (25) is not observable on \( \tau \). Consequently, if the LTV system (25) is observable on \( \tau \), then (4) is true.

To show that (4) is also a sufficient condition, suppose first that \( c_1 \neq 0 \). For \( \tau = t_0 \) one has \( f(t_0) = \begin{bmatrix} 1 - d(t_0)d^T(t_0) \end{bmatrix} c_1 \).

If \( f(t_0) > 0 \), then it follows, using Proposition 1, that \( c^T W_0(t_0, t_1) c > 0 \). Otherwise, it must be \( c_1 = c_1 d(t_0) \) for some \( c_1 \neq 0 \). Suppose then that \( c_1 = c_1 d(t_0) \). Then,
\[
\begin{align*}
\frac{d}{d\tau} f(t) &= -c_1d(t_0) + \begin{bmatrix} 1 - d(t_0)d^T(t_0) \end{bmatrix} c_2.
\end{align*}
\]

Consider first \( c_1 = c_1 d(t_0), c_2 = 0 \). Then, using (4), it is possible to conclude that
\[ \|f(t_1)\| = \|c_1 [1 - d(t_1)d^T(t_1)] d(t_0)\| > 0 \]
and, using Proposition 1, it follows that \( c^T W_0(t_0, t_1) c > 0 \) for \( c_1 = c_1 d(t_0) \) and \( c_2 = 0 \). Suppose now that \( c_1 = c_1 d(t_0) \) and \( c_2 = c_2 d(t_0) \). Then
\[ \|f(t_1)\| = \|c_1 + (t_1 - t_0)c_2||[1 - d(t_1)d^T(t_1)] d(t_0)||. \]

If \( c_1 + (t_1 - t_0)c_2 \neq 0 \), then it is possible to conclude, from (4), that \( \|f(t_1)\| > 0 \) and, using Proposition 1, \( c^T W_0(t_0, t_1) c > 0 \). Otherwise, \( c_1 + (t_1 - t_0)c_2 = 0 \), there exists, by continuity, \( t_0 < t_2 < t_1 \) such that \( d^T(t_0) d(t_2) < 1 \). As such
\[ \|f(t_1)\| = \|c_1 + (t_2 - t_0)c_2||[1 - d(t_2)d^T(t_2)] d(t_0)||. \]

It is positive. Again, using Proposition 1, it follows that \( c^T W_0(t_0, t_1) c > 0 \) for \( c_1 = c_1 d(t_0) \) and \( c_2 = c_2 d(t_0) \). If \( c_2 = c_2 d(t_0) \), with \( d(t_0) = 0 \), there exists \( e > 0 \) such that
\[ f(t_0 + e) = c_1 [1 - d(t_0 + e)d^T(t_0 + e)] d(t_0) + ec_1 [1 - d(t_0 + e)d^T(t_0 + e)] d(t_0) \]
and \( d(t_0 + e) \) cannot be expressed as a linear combination of \( d(t_0) \) and \( d(t_0) \). As such, \( f(t_0 + e) > 0 \) and, using Proposition 1, \( c^T W_0(t_0, t_1) c > 0 \) for \( c_1 = c_1 d(t_0) \) and \( c_2 = c_2 d(t_0) \). This allows to conclude, so far, that if \( c_1 \neq 0 \), then \( c^T W_0(t_0, t_1) c > 0 \). Suppose now that \( c_1 = 0 \), which implies that \( \|c_2\| = 1 \). Then, \( f(t, 0) = 0 \) and
\[ \frac{d}{d\tau} f(t) = [1 - d(t)d^T(t)] c_2. \]
If \( c_2 \neq \pm d(t_0) \), then
\[ \left\| \frac{d}{d\tau} f(t) \right\|_{t_0} > 0 \]
and using Proposition 1 twice, it is possible to conclude that \( c^T W_0(t_0, t_1) c > 0 \). Otherwise, if \( c_2 = \pm d(t_0) \) then
\[ \|f(t_1)\| = \|t_1 - t_0\||[1 - d(t_1)d^T(t_1)] d(t_0)|| > 0. \]

Again, using Proposition 1, it is possible to conclude that \( c^T W_0(t_0, t_1) c > 0 \) for all \( \|c\| = 1 \), which means that (25) is observable. \( \square \)

**Theorem 8.** The LTV system (25) is uniformly completely observable if and only if
\[ \exists t > 0 \quad \forall \tau > 0 \quad \int_{t}^{t+\delta} [d^T(t) d(t)]^2 dr \leq \delta (1 - \alpha). \]
(26)

**Proof.** The proof of sufficiency follows similar steps to Theorem 7 considering uniformity bounds that stem from the persistent excitation condition (26). Therefore it is omitted. To show that (26) is also necessary, suppose that (26) does not hold. Then,
\[ \exists t > 0 \quad \forall \tau > 0 \quad \int_{t}^{t+\delta} [d^T(t) d(t)]^2 dr \geq \delta (1 - \alpha). \]
(27)

Let \( c = \begin{bmatrix} d^T(t) & 0 \end{bmatrix} \in \mathbb{R}^6 \). Then, it is easily shown that
\[ c^T W_0(t^*, t^* + \delta) c = \int_{t^*}^{t^*+\delta} \left(1 - [d^T(t) d(t)]^2\right) dt. \]
(28)
Using (27) in (28) allows to conclude that, for all \( \alpha > 0 \) and \( \delta > 0 \) there exists a time instant \( t^* \geq t_0 \) such that \( c^T W (t^*, t^* + \delta) c < \delta \alpha \), which means that the LTV system (25) is not uniformly completely observable. Therefore, if the LTV system (25) is uniformly completely observable, (25) is true.

**Remark 2.** Notice that (26) is true if and only if (12) is true. The former was preferred in this section because it simplifies the proof of Theorem 8.

6. Simulation results

This section presents realistic simulation results for the source localization problem in order to evaluate the performance achieved with the proposed solutions. Further testing revealed that similar performances are achieved for the navigation problem based on direction measurements.

In the simulations, the source and the agent trajectories are those depicted in Fig. 1. Clearly, the persistent excitation condition (12) is satisfied, which allows to apply the solutions proposed in the paper. The drift velocity of the source was set to \( \mathbf{v}_s(t) = [1 0 0]^T \text{ (m/s)} \), while the drift velocity of the agent was set to \( \mathbf{v}_c(t) = [-0.5 0 0]^T \text{ (m/s)} \), which gives \( \mathbf{v}_{sc}(t) = [1.5 0 0]^T \text{ (m/s)} \) for the relative drift velocity.

Noise was considered for both the direction measurements and the relative velocity of the agent \( \mathbf{v}_c(t) \). In particular, additive zero mean white Gaussian noise was considered for \( \mathbf{v}_c(t) \), with standard deviation of 0.01 m/s, while the direction readings were assumed perturbed by rotations about random vectors of an angle modeled by zero-mean white Gaussian noise, with standard deviation of 1°. The Kalman filter parameters were set to \( \Xi = \text{diag} (10^{-2}I, 10^{-3}I, 10^{-2}) \) and \( \Theta = I \). The initial estimates were all set to zero.

The evolution of the estimation error is shown in Fig. 2. As it is possible to see, the initial transients due to the mismatch of the initial conditions quickly fade out, resulting in state estimates very close to the true value.

In order to better evaluate the performance of the proposed solution, the Monte Carlo method was applied. The simulation was carried out 1000 times with different, randomly generated noise signals. The mean and standard deviation were computed for each simulation and averaged over the 1000 simulations. The results are depicted in Table 1, where the results obtained with an Extended Kalman Filter with similar parameters are also presented. As the initial estimate for the source location cannot be set to zero with the EKF (in the linearization there appear terms divided by the norm of this estimate), the initial source position estimate was set to \([100]^T\text{ m}\). The convergence speed results slightly smaller. As it is possible to observe, the proposed solutions achieve similar performance to the EKF, while providing, at the same time, global exponential stability guarantees.

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**Fig. 1.** Trajectory described by the agent and the source.

**Fig. 2.** Evolution of the estimation errors.
Table 1

<table>
<thead>
<tr>
<th>Proposed solution</th>
<th>EKF</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{x_1}$ (m)</td>
<td>$8.5 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\sigma_{x_2}$ (m)</td>
<td>$3.3 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\sigma_{x_3}$ (m)</td>
<td>$1.2 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\sigma_{v_x}$ (m/s)</td>
<td>$4.8 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\sigma_{v_y}$ (m/s)</td>
<td>$4.8 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\sigma_{v_z}$ (m)</td>
<td>$1.1 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

7. Conclusions

This paper presented a set of globally exponentially stable Kalman filters for the problems of source localization and navigation based on direction measurements to a single source. The observability of the systems was fully characterized, which allowed to conclude about the asymptotic stability of the Kalman filters. The observability conditions that were derived are directly related to the motion of the agent/vehicle and as such they are useful for motion planning and control. Simulation results were presented that illustrate the good performance achieved by the proposed solutions, which were also compared with the EKF, achieving similar performance but with global asymptotic stability guarantees. Future work includes the extension of the present work to the case where directions to multiple sources are available for navigation purposes.

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References