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GES Source Localization and Navigation based on Discrete-Time Bearing Measurements

Pedro Batista, Carlos Silvestre, and Paulo Oliveira

Abstract— **This paper addresses the problems of source localization and navigation based on discrete-time single direction measurements, in 3-D, in addition to relative velocity readings. For source localization, an agent aims at estimating the position of a source relative to itself, while for navigation the agents aims to estimate its own position assuming that it also has access to the position of the source in an inertial frame. Additionally, unknown constant drift velocities are considered and explicitly estimated. The design follows essentially by considering an augmented system, which is linear, and thoroughly address its observability and its relation with the original nonlinear system. The final estimation solution is a Kalman filter, with globally exponentially stable (GES) error dynamics. Simulation results are presented that illustrate the achievable performance with the proposed solution.**

I. Introduction

A recurring problem that agents face in robotics is that of estimating the position of an external target (which can be another agent), either in absolute or relative coordinates. This problem is often denominated as that of source localization, when there is a device in the target that emits signals (hence the designation as source), although sometimes it is also referred to as target localization, more often in warfare applications. In mobile robotics this can be the case of an unmanned ground vehicle attempting to estimate the position of a beacon, while in aerial robotics it can be an unmanned aerial vehicle trying to estimate the position of another unmanned aerial vehicle. Parallel to the problem of source localization is that of agent navigation, often navigation of autonomous vehicles. In this case the aim is to estimate the position of the agent itself, among other variables, assuming known positions of externals features of devices.

Earlier solutions to the problem of navigation resort to absolute position measurements. The celebrated Global Positioning System (GPS) is usually the workhorse in open-space mission, while Long Baseline (LBL) acoustic positioning system are often employed for underwater scenarios. Driven by the problem of source localization, other solutions have been pursued for both problems based on different sensors, such as the use of single range measurements, see e.g. [1], [2], [3], [4], [5], [6], [7], and references therein. The duality between navigation and source localization is evidenced in [8].

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The authors are with the Institute for Systems and Robotics, Laboratory of Robotics and Systems in Engineering and Science, Portugal. Pedro Batista is also with Instituto Superior Técnico, Universidade de Lisboa, Portugal. Carlos Silvestre is also with the Department of Electrical and Computer Engineering, Faculty of Science and Technology of the University of Macau. Paulo Oliveira is also with the Department of Mechanical Engineering, Instituto Superior Técnico, Universidade de Lisboa, Portugal.
{pbatista,cjs,pjcro}@isr.ist.utl.pt

An alternative to single range measurements is the use of bearing measurements, see e.g. [9], [10], [11], [12], and references therein. In previous work by the authors [13] the problems of source localization and navigation based on bearing measurements were addressed in a continuous-time framework, where the duality between both problems is again evidenced. In practice, the bearing measurements are often acquired in discrete-time, which poses challenge both in terms of observability analysis and filter design. The problems of navigation and source localization are revisited in this paper, building on the results obtained in [13], considering discrete-time bearing measurements. A discrete-time augmented system is derived, which is linear time-varying (LTV), and its observability analyzed, in a constructive manner, such that the design of an observer (or filter) follows naturally using estimation tools for linear systems. A Kalman filter is proposed with globally exponentially stable (GES) error dynamics.

The paper is organized as follows. The problems considered in the paper and the nominal system dynamics are introduced in Section II. The topic of source localization is addressed in Section III in detail, while in Section IV the dual for navigation is presented building on the results for source localization. Simulation results are discussed in Section V and Section VI summarizes the main results of the paper.

A. Notation

Throughout the paper the symbols $\mathbf{0}$ and $\mathbf{I}$ denote a matrix of zeros and the identity matrix, respectively, while $\operatorname{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$ is a block diagonal matrix. For $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{x} \times \mathbf{y}$ represent the inner and cross products, respectively.

II. Problem statements

A. Source localization

In order to set the problem framework, let $\{I\}$ denote an inertial frame and consider an agent, whose inertial position at time $t = t_k$, $t_k = t_0 + kT$, $k = 1, 2, \ldots$, $T > 0$, is denoted as $\mathbf{p}(k) \in \mathbb{R}^3$, aiming to determine the position of a source relative to the agent. Further define a coordinate frame $\{B\}$ attached to the agent, usually denominated as the body-fixed reference frame. Let the inertial position of the source at time $t = t_k$ be denoted by $\mathbf{s}(k) \in \mathbb{R}^3$ and assume that the source is drifting with constant inertial unknown velocity $\mathbf{v}_s(k) \in \mathbb{R}^3$. Suppose that the agent has access to its velocity relative to the fluid expressed in body-fixed coordinates, $\mathbf{v}_r(t) \in \mathbb{R}^3$, as well as to its attitude, in the form of a rotation matrix $\mathbf{R}(t) \in SO(3)$, from body-fixed coordinates to inertial coordinates. The fluid is assumed to have constant velocity in inertial coordinates $\mathbf{v}_f(k) \in \mathbb{R}^3$.

Finally, assume that the agent measures the direction of the source relative to itself, given by

$$\mathbf{d}(k) = \frac{\mathbf{s}(k) - \mathbf{p}(k)}{\|\mathbf{s}(k) - \mathbf{p}(k)\|} \in S(2). \tag{1}$$

The problem of source localization considered in this paper is that of estimating the position of the source relative to the agent, $\mathbf{s}(k) - \mathbf{p}(k)$, as well as their relative drifting velocities, $\mathbf{v}_s(k) - \mathbf{v}_f(k)$, based on the agent relative velocity measurements, $\mathbf{v}_r(t)$, and the direction measurements, $\mathbf{d}(k)$.

The evolution of the position of the source is simply given by $\mathbf{s}(k+1) = \mathbf{s}(k) + T\mathbf{v}_s(k)$, while the evolution of the position of the agent can be written as

$$\mathbf{p}(k+1) = \mathbf{p}(k) + T\mathbf{v}_f(k) + \mathbf{u}(k), \tag{2}$$

with $\mathbf{u}(k) := \int_{t_k}^{t_{k+1}} \mathbf{R}(\tau)\mathbf{v}_r(\tau)\,d\tau$. Let

$$\mathbf{r}(k) := \mathbf{s}(k) - \mathbf{p}(k)$$

and

$$\mathbf{v}_{sa}(k) := \mathbf{v}_s(k) - \mathbf{v}_f(k)$$

denote the position of the source relative to the agent and the relative drift velocities, respectively. Then, the discrete-time system for source localization can be written as

$$\begin{cases} \mathbf{r}(k+1) = \mathbf{r}(k) + T\mathbf{v}_{sa}(k) - \mathbf{u}(k) \\ \mathbf{v}_{sa}(k+1) = \mathbf{v}_{sa}(k) \\ \mathbf{d}(k+1) = \frac{\mathbf{r}(k+1)}{\|\mathbf{r}(k+1)\|} \end{cases}. \tag{3}$$

In other words, the problem of source localization considered in the paper is that of designing an estimator for the nonlinear system (3) with globally exponentially stable error dynamics.

B. Navigation

For the problem of navigation, the inertial position of the source is assumed available to the agent, at the same rate of the bearing measurements. In addition, the velocity of the source needs not be constant and its knowledge is not required for the agent. The corresponding discrete-time system is

$$\begin{cases} \mathbf{p}(k+1) = \mathbf{p}(k) + T\mathbf{v}_f(k) + \mathbf{u}(k) \\ \mathbf{v}_f(k+1) = \mathbf{v}_f(k) \\ \mathbf{d}(k+1) = \frac{\mathbf{s}(k+1) - \mathbf{p}(k+1)}{\|\mathbf{s}(k+1) - \mathbf{p}(k+1)\|} \end{cases}. \tag{4}$$

The problem of navigation based on bearing measurements to a single source considered in the paper is that of designing an estimator for the nonlinear system (4) with globally exponentially stable error dynamics.

The following (mild) assumption is considered throughout the paper.

Assumption 1: The movement of the agent and the source is such that the direction measurements satisfy

$$\mathbf{d}(k) \cdot \mathbf{d}(k+1) > 0$$

for all $k \geq k_0$.

III. SOURCE LOCALIZATION FILTER DESIGN

In previous work by the authors [13] the problem of source localization based on direction measurements was addressed considering a continuous framework. In the proposed approach, the distance from the agent to the source was considered as a system state and the output of the system was redefined so that it could be considered as linear. This paper follows up this approach but providing an exact solution is discrete-time, considering discrete-time bearing measurements. While the estimator dynamics are in discrete-time, the final filtering solution provides estimates in continuous-time using the relative velocity readings of the agent, which are available at high rates.

A. System augmentation

Define as system states

$$\begin{cases} \mathbf{x}_1(k) := \mathbf{r}(k) \\ \mathbf{x}_2(k) := \mathbf{v}_{sa}(k) \\ x_3(k) := \|\mathbf{r}(k)\| \end{cases}.$$

From (3) the evolution of the first two states is simply

$$\begin{cases} \mathbf{x}_1(k+1) = \mathbf{x}_1(k) + T\mathbf{x}_2(k) - \mathbf{u}(k) \\ \mathbf{x}_2(k+1) = \mathbf{x}_2(k) \end{cases}. \tag{5}$$

In order to describe the evolution of $x_3(k)$, notice that, using the equation of the bearing measurements (1), it is possible to write

$$\mathbf{d}(k+1)\,x_3(k+1) = \mathbf{x}_1(k+1). \tag{6}$$

Computing the inner product of both sides of (6) with $\mathbf{d}(k+1)$ gives

$$x_3(k+1) = \mathbf{d}(k+1) \cdot \mathbf{x}_1(k+1). \tag{7}$$

Substituting (5) in (7) gives

$$\begin{aligned} x_3(k+1) &= \mathbf{d}(k+1) \cdot \mathbf{x}_1(k) + T\mathbf{d}(k+1) \cdot \mathbf{x}_2(k) \\ &\quad - \mathbf{d}(k+1) \cdot \mathbf{u}(k). \end{aligned} \tag{8}$$

This form of evolution for $x_3(k)$ is undesirable because $x_3(k+1)$ does not depend on $x_3(k)$. In order to make that happen, notice again that $\mathbf{x}_1(k) = \mathbf{d}(k)\,x_3(k)$, which allows to rewrite (8) as

$$\begin{aligned} x_3(k+1) &= T\mathbf{d}(k+1) \cdot \mathbf{x}_2(k) + \mathbf{d}(k+1) \cdot \mathbf{d}(k)\,x_3(k) \\ &\quad - \mathbf{d}(k+1) \cdot \mathbf{u}(k). \end{aligned}$$

Define the augmented state vector

$$\mathbf{x}(k) := \begin{bmatrix} \mathbf{x}_1(k) \\ \mathbf{x}_2(k) \\ x_3(k) \end{bmatrix} \in \mathbb{R}^{3+3+1}.$$

From (6) one has

$$\mathbf{x}_1(k+1) - \mathbf{d}(k+1)\,x_3(k+1) = \mathbf{0}. \tag{9}$$

Considering (9) and discarding the original nonlinear output (1) allows to write the discrete-time system

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k) \\ \mathbf{y}(k+1) = \mathbf{C}(k+1)\mathbf{x}(k+1) \end{cases}, \tag{10}$$

where

$$\mathbf{A}(k) := \begin{bmatrix} \mathbf{I} & T\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & T\mathbf{d}^T(k+1) & \mathbf{d}(k+1) \cdot \mathbf{d}(k) \end{bmatrix} \in \mathbb{R}^{7\times 7},$$

$$\mathbf{B}(k) := \begin{bmatrix} -\mathbf{I} \\ \mathbf{0} \\ -\mathbf{d}^T(k+1) \end{bmatrix} \in \mathbb{R}^{7\times 3},$$

and $\mathbf{C}(k) := \begin{bmatrix} \mathbf{I} & \mathbf{0} & -\mathbf{d}(k) \end{bmatrix} \in \mathbb{R}^{3\times 7}$.

B. Observability analysis

The following result addresses the observability of the discrete-time linear system (10).

Theorem 1: Under Assumption 1, the discrete-time linear system (10) is observable on $[k, k+3]$, for a fixed $k \geq k_0$, if and only if

$$\begin{aligned} &2\,[\mathbf{d}(k)\cdot\mathbf{d}(k+1)][\mathbf{d}(k)\cdot\mathbf{d}(k+2)][\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)] \\ &\quad +1 - [\mathbf{d}(k)\cdot\mathbf{d}(k+1)]^2 - [\mathbf{d}(k)\cdot\mathbf{d}(k+2)]^2 \\ &\quad - [\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)]^2 \neq 0. \end{aligned} \tag{11}$$

Proof: The proof resorts to the analysis of the observability matrix $\mathcal{O}(k,k+3)$ associated with the pair $(\mathbf{A}(k),\mathbf{C})$ on $[k,k+3]$, given by

$$\mathcal{O}(k,k+3)=\begin{bmatrix}\mathbf{C}(k)\\ \mathbf{C}(k+1)\mathbf{A}(k)\\ \mathbf{C}(k+1)\mathbf{A}(k+1)\mathbf{A}(k)\end{bmatrix}\in\mathbb{R}^{9\times 7}.$$

The discrete time-varying linear system (10) is observable on $[k,k+3]$ if and only if the observability matrix $\mathcal{O}(k,k+3)$ is full rank. Let $\mathbf{c}=\left[\mathbf{c}_1^T\,\mathbf{c}_2^T\,c_3\right]^T\in\mathbb{R}^7$, $\mathbf{c}_1,\mathbf{c}_2\in\mathbb{R}^3$, $c_3\in\mathbb{R}$. It is a simple matter of computation to show that

$$\mathbf{C}(k)\mathbf{c}=\mathbf{c}_1-c_3\mathbf{d}(k),\tag{12}$$

$$\mathbf{C}(k+1)\mathbf{A}(k)\mathbf{c}=\mathbf{c}_1+T\left[\mathbf{I}-\mathbf{d}(k+1)\mathbf{d}^T(k+1)\right]\mathbf{c}_2 -c_3\mathbf{d}(k+1)\cdot\mathbf{d}(k)\mathbf{d}(k+1),\tag{13}$$

and

$$\begin{aligned}\mathbf{C}(k+2)\mathbf{A}(k+1)\mathbf{A}(k)\mathbf{c}&=\mathbf{c}_1\\ &+T\left[2\mathbf{I}-\mathbf{d}(k+2)\mathbf{d}^T(k+2)\right]\mathbf{c}_2\\ &-T\mathbf{d}(k+2)\cdot\mathbf{d}(k+1)\mathbf{d}(k+1)\cdot\mathbf{c}_2\mathbf{d}(k+2)\\ &-c_3\mathbf{d}(k)\cdot\mathbf{d}(k+1)\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)\mathbf{d}(k+2).\end{aligned}\tag{14}$$

To prove necessity, suppose that (11) does not hold, i.e.,

$$\begin{aligned}&2\left[\mathbf{d}(k)\cdot\mathbf{d}(k+1)\right]\left[\mathbf{d}(k)\cdot\mathbf{d}(k+2)\right]\left[\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)\right]\\ &+1-\left[\mathbf{d}(k)\cdot\mathbf{d}(k+1)\right]^2-\left[\mathbf{d}(k)\cdot\mathbf{d}(k+2)\right]^2\\ &-\left[\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)\right]^2=0,\end{aligned}\tag{15}$$

and consider two different cases: i) $\mathbf{d}(k+1)=\mathbf{d}(k+2)$; and ii) $\mathbf{d}(k+1)\neq\mathbf{d}(k+2)$. In the first case, with $\mathbf{d}(k+1)=\mathbf{d}(k+2)$, notice that (15) is always satisfied regardless of $\mathbf{d}(k)$. Let $\mathbf{c}_1=\mathbf{0}$, $\mathbf{c}_2=\mathbf{d}(k+1)$, and $c_3=0$. Then, substituting $\mathbf{c}$ in (12)-(14) gives $\mathcal{O}(k,k+3)\mathbf{c}=\mathbf{0}$, which means that the observability matrix is not full rank and hence the LTV system (10) is not observable. In the second case, with $\mathbf{d}(k+1)\neq\mathbf{d}(k+2)$, let $\mathbf{c}_1=c_3\mathbf{d}(k)$ and

$$\mathbf{c}_2=\alpha\frac{c_3}{2T}\mathbf{d}(k+1)-\frac{c_3}{T}\left[\mathbf{d}(k)-\mathbf{d}(k)\cdot\mathbf{d}(k+1)\mathbf{d}(k+1)\right],$$

with

$$\begin{aligned}\alpha\;:=\;&\frac{2\left[\mathbf{d}(k)\cdot\mathbf{d}(k+1)\right]\left[\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)\right]^2}{1-\left[\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)\right]^2}\\ &-\frac{\left[\mathbf{d}(k)\cdot\mathbf{d}(k+2)\right]\left[\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)\right]}{1-\left[\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)\right]^2}\\ &-\frac{\left[\mathbf{d}(k)\cdot\mathbf{d}(k+1)\right]}{1-\left[\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)\right]^2},\end{aligned}\tag{16}$$

and $c_3\neq 0$ such that $\mathbf{c}$ is unit vector. Again, tedious but straightforward computations, substituting $\mathbf{c}$ in (12)-(14) and using (15), allow to show that $\mathcal{O}(k,k+3)\mathbf{c}=\mathbf{0}$, which means that the observability matrix is not full rank and hence the LTV system (10) is not observable. Thus, it has been shown that if (11) does not hold, the linear time-varying system (10) is not observable on $[k,k+3]$. Hence, if the linear time-varying system (10) is observable on $[k,k+3]$ then (11) must hold, thus concluding the proof of necessity.

In order to show sufficiency, suppose that the system is not observable, which means that there exists a unit vector $\mathbf{c}$ such that $\mathcal{O}(k,k+3)\mathbf{c}=\mathbf{0}$. From (12) it must be

$$\mathbf{c}_1=c_3\mathbf{d}(k).\tag{17}$$

Consider first that $c_3=0$. Then, from (17) it must be also $\mathbf{c}_1=\mathbf{0}$. Substituting that in (13) allows to conclude that it must be $\mathbf{c}_2=\pm\mathbf{d}(k+1)$. Substituting $\mathbf{c}_1=\mathbf{0}$, $\mathbf{c}_2=\pm\mathbf{d}(k+1)$, and $c_3=0$ in (14) gives

$$\mathbf{d}(k+1)=\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)\mathbf{d}(k+2),$$

whose only solution, under Assumption 1, is $\mathbf{d}(k+1)=\mathbf{d}(k+2)$. With $\mathbf{d}(k+1)=\mathbf{d}(k+2)$ it follows that (15) is true. Hence, thus far it has been shown that if a unit vector $\mathbf{c}$ exists, with $c_3=0$, such that $\mathcal{O}(k,k+3)\mathbf{c}=\mathbf{0}$, then (11) cannot hold. Consider now $c_3\neq 0$ and substitute (17) in (13), which gives

$$\begin{aligned}&c_3\left[\mathbf{d}(k)-\mathbf{d}(k+1)\cdot\mathbf{d}(k)\mathbf{d}(k+1)\right]\\ &+T\left[\mathbf{I}-\mathbf{d}(k+1)\mathbf{d}^T(k+1)\right]\mathbf{c}_2=\mathbf{0}.\end{aligned}\tag{18}$$

Decompose $\mathbf{c}_2$ as

$$\mathbf{c}_2=\frac{\beta}{2T}\mathbf{d}(k+1)+\mathbf{c}_2',\tag{19}$$

where $\beta\in\mathbb{R}$ and $\mathbf{c}_2'\in\mathbb{R}^3$ is orthogonal to $\mathbf{d}(k+1)$. Substituting (19) in (18) implies

$$\mathbf{c}_2'=-\frac{c_3}{T}\left[\mathbf{d}(k)-\mathbf{d}(k+1)\cdot\mathbf{d}(k)\mathbf{d}(k+1)\right],$$

meaning that it must be

$$\mathbf{c}_2=\frac{\beta}{2T}\mathbf{d}(k+1)-\frac{c_3}{T}\left[\mathbf{d}(k)-\mathbf{d}(k+1)\cdot\mathbf{d}(k)\mathbf{d}(k+1)\right]\tag{20}$$

for some $\beta\in\mathbb{R}$. Substituting (17) and (20) in (14) and simplifying gives

$$\begin{aligned}&-c_3\mathbf{d}(k)+\beta\mathbf{d}(k+1)-\beta\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)\,\mathbf{d}(k+2)\\ &+c_3\mathbf{d}(k)\cdot\mathbf{d}(k+2)\,\mathbf{d}(k+2)+2c_3\mathbf{d}(k)\cdot\mathbf{d}(k+1)\,\mathbf{d}(k+1)\\ &-2c_3\mathbf{d}(k)\cdot\mathbf{d}(k+1)\,\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)\,\mathbf{d}(k+2)=\mathbf{0}.\end{aligned}\tag{21}$$

Notice that (21) is a sum of terms along three directions, $\mathbf{d}(k)$, $\mathbf{d}(k+1)$, and $\mathbf{d}(k+2)$. Hence (21) is satisfied if and only if the inner product of the left side of (21) with these three directions is null. It is easy to verify that the inner product of the left side of (21) with $\mathbf{d}(k+2)$ is always null, regardless of β. Computing the inner product of both sides of (21) with $\mathbf{d}(k+1)$ allows to write

$$\begin{aligned}&\beta\left(1-\left[\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)\right]^2\right)=\\ &-c_3\left[\mathbf{d}(k)\cdot\mathbf{d}(k+1)+\mathbf{d}(k)\cdot\mathbf{d}(k+2)\,\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)\right]\\ &+2c_3\mathbf{d}(k)\cdot\mathbf{d}(k+1)\left[\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)\right]^2.\end{aligned}\tag{22}$$

Suppose first that $\mathbf{d}(k+1)\neq\mathbf{d}(k+2)$, which implies under Assumption 1 that $\left[\mathbf{d}(k+1)\cdot\mathbf{d}(k+2)\right]^2\neq 1$. Then, it follows from (22) that it must be

$$\beta=c_3\alpha,\tag{23}$$

with α as defined in (16). Substituting (23) in (21), computing the inner product of both sides of (21) with $\mathbf{d}(k)$, and simplifying allows to conclude that (15) holds. On the other hand, if $\mathbf{d}(k+1)=\mathbf{d}(k+2)$, then (15) also holds. Thus, it has been shown that if a unit vector $\mathbf{c}$ exists, with $c_3\neq 0$, such that $\mathcal{O}(k,k+3)\mathbf{c}=\mathbf{0}$, then (11) cannot hold. But that had already been show for $c_3=0$, which allows to conclude that if a unit vector $\mathbf{c}$ exists such that $\mathcal{O}(k,k+3)\mathbf{c}=\mathbf{0}$ or, equivalently, if the LTV system (10) is not observable, then (11) cannot be true. Thus, if (11) holds, the LTV system (10) is observable, thus concluding the proof of sufficiency. ∎

Although given the evolution of the direction readings the condition (11) can be easily verified, it is rather obscure in terms of interpretation. The following lemma sheds light into this issue.

Lemma 1: The condition (11) is satisfied if and only if the set of vectors $\mathcal{D}:=\{\mathbf{d}(k),\mathbf{d}(k+1),\mathbf{d}(k+2)\}$ is linearly independent.

Proof: Suppose first that the set of vectors $\mathcal{D}$ is not linearly independent. Then, one of the vectors of the set can be written as a linear combination of the other two. Due to the symmetry of (11) and the fact that all the vectors of $\mathcal{D}$ are unit vectors, suppose, without loss of generality, that it is possible to write

$$\mathbf{d}(k{+}2) = c_0\mathbf{d}(k) + c_1\mathbf{d}(k{+}1) \tag{24}$$

for some $[c_0\ c_1]^T \neq \mathbf{0}$. Notice that, as all vectors of $\mathcal{D}$ are unit vectors, it follows from (24) that

$$c_0^2 + c_1^2 + c_0c_1\mathbf{d}(k)\cdot\mathbf{d}(k{+}1) = 1. \tag{25}$$

Substituting (24) in (15) and using (25) allows to conclude that (11) is not verified. By contraposition, if (11) holds, then the set of vectors $\mathcal{D}$ must be linearly independent.

Suppose now that the set of vectors $\mathcal{D}$ is linearly independent. Then, there must exits $c_0 \in \mathbb{R}$, $c_1 \in \mathbb{R}$, and $c_2 \in \mathbb{R}$, with $c_2 \neq 0$, such that

$$\mathbf{d}(k{+}2) = c_0\mathbf{d}(k) + c_1\mathbf{d}(k{+}1) + c_2\frac{\mathbf{d}(k)\times\mathbf{d}(k{+}1)}{\|\mathbf{d}(k)\times\mathbf{d}(k{+}1)\|}. \tag{26}$$

From (26), as all vectors of $\mathcal{D}$ are unit vectors, and as $\mathbf{d}(k)\times\mathbf{d}(k{+}1)$ is orthogonal to both $\mathbf{d}(k)$ and $\mathbf{d}(k{+}1)$, it follows that

$$c_0^2 + c_1^2 + c_2^2 + c_0c_1\mathbf{d}(k)\cdot\mathbf{d}(k{+}1) = 1. \tag{27}$$

Now, using (26) and (27) allows to write

$$\begin{aligned} &2\left[\mathbf{d}(k)\cdot\mathbf{d}(k{+}1)\right]\left[\mathbf{d}(k)\cdot\mathbf{d}(k{+}2)\right]\left[\mathbf{d}(k{+}1)\cdot\mathbf{d}(k{+}2)\right]\\ &\quad+1-\left[\mathbf{d}(k)\cdot\mathbf{d}(k{+}1)\right]^2-\left[\mathbf{d}(k)\cdot\mathbf{d}(k{+}2)\right]^2\\ &-\left[\mathbf{d}(k{+}1)\cdot\mathbf{d}(k{+}2)\right]^2 = c_2^2\left[1-\left[\mathbf{d}(k)\cdot\mathbf{d}(k{+}1)\right]^2\right].\end{aligned} \tag{28}$$

As the set of vectors $\mathcal{D}$ is linearly independent, it must be $[\mathbf{d}(k)\cdot\mathbf{d}(k{+}1)]^2 \neq 1$. In addition, $c_2 \neq 0$. Hence, it follows from (28) that (11) holds. This concludes the proof. ■

Before proceeding, it is important to remark that in the definition of the augmented system (10) the original nonlinear output was discarded. In addition, there is nothing in (10) imposing that $x_3(k) = \|\mathbf{x}_1(k)\|$. As such, care must be taken about the conclusions drawn for the original system (3) from the conclusions derived for (10). The following theorem addresses this issue.

Theorem 2: Consider Assumption 1 and suppose that (11) holds. Then:

i) the initial condition of (10) corresponds to the initial condition of (3), i.e.,

$$\begin{cases} \mathbf{x}_1(k_0) = \mathbf{r}(k_0) \\ \mathbf{x}_2(k_0) = \mathbf{v}_{sa}(k_0) \\ x_3(k_0) = \|\mathbf{r}(k_0)\| \end{cases}; \tag{29}$$

ii) the nonlinear system (3) is observable in the sense that, given the system input $\mathbf{u}(k)$ and output $\mathbf{d}(k)$, for $k = k_0$, k_0+1, $k = k_0+2$, its initial condition is uniquely determined; and

iii) an observer for the linear system (10) with globally exponentially stable error dynamics is also an observer for the nonlinear system (3), with globally exponentially stable error dynamics.

Proof: Due to space limitations, only a sketch of the proof is provided. Under the terms of Theorem 1, the initial condition of the LTV system (10) is uniquely determined by the corresponding system output and input. The proof of the first part of the theorem follows by proving that (29) explains the system output, which is null for the present system, according to (9). As the initial condition is uniquely determined, if (29) explains the output of the system, it must correspond to the initial condition. As it is unique and corresponds to that of the nonlinear system (3), the second part of the theorem immediately follows. The proof of the last part follows from the first two. Indeed, the estimates of an observer for (10) with globally exponentially stable error dynamics approach the true state globally exponentially fast. But as it has been shown that this corresponds to the state of the nonlinear system (3), it follows that those estimates approach the state of (3) globally exponentially fast, thus concluding the proof. ■

The observability conditions that were derived in this section consider the smallest possible interval for observability. However, less demanding conditions may be derived for larger intervals. In particular, considering larger intervals (more measurements), the directions measurements need only span $\mathbb{R}^2$ for the system to be observable. Unfortunately, space limitations prevent the derivation of such results in the paper and are left for future work. These results follow from the analysis of the observability matrix for larger intervals, e.g. the observability matrix $\mathcal{O}(k, k+4)$ instead of $\mathcal{O}(k, k+3)$. This is an interesting behavior that is not present in the continuous-time case.

C. Kalman filter and further discussion

The design of an observer for (10) may follow using a myriad of tools for linear systems, e.g., the Luenberger observer as detailed in [14, Theorem 29.2], which would allow to choose the convergence rate. In this paper, the Kalman filter was chosen as the estimation solution.

In order to ensure stability of the Kalman filter, stronger forms of observability are required as this is a time-varying system, in particular, uniform complete observability. For the sake of ease of presentation, this paper focuses on the derivation of observability conditions. The conditions for uniform complete observability, which are somehow related to the observability of the system but considering uniform bounds in time, will be derived in an extended version of this paper.

It is important to stress that, in spite of the fact that, in nominal terms, the drift velocity was assumed constant, it is possible to consider, during the design of the Kalman filter, that this state is driven by a white Gaussian process, with zero mean. By appropriate adjustment of the magnitude of the corresponding filter parameter (state disturbance variance), it is possible to allow the filter to estimate slowly time-varying source velocities. It is also important to mention that it is not claimed that the solution is optimal, as there exists multiplicative noise.

In the proposed setup, it is assumed that the agent measures its relative velocity. An alternative setting is trivially derived from this using position measurements instead and estimating the source position directly, as well as its drift velocity. Using the relative velocity allows to obtain estimates at a higher rate resorting to open-loop integration between discrete-time bearing measurements, with no loss of stability, as given by

$$\begin{cases} \hat{\mathbf{r}}(t) = \hat{\mathbf{r}}(t_k) + (t-t_k)\,\hat{\mathbf{v}}_{sa}(t_k) - \int_{t_k}^{t}\mathbf{R}(\tau)\,\mathbf{v}_r(\tau)\,d\tau \\ \hat{\mathbf{v}}_c(t) = \hat{\mathbf{v}}_c(t_k) \end{cases}$$

for $t_k < t < t_{k+1}$.

IV. Navigation filter design

Define as system states

$$\left\{\begin{array}{l} \boldsymbol{x}_1(k) := \mathbf{p}\,(k) \\ \boldsymbol{x}_2(k) := \mathbf{v}_f\,(k) \\ x_3\,(k) := \left\|\mathbf{s}\,(k) - \boldsymbol{x}_1\,(k)\right\| \end{array}\right. .$$

From the equation of the direction measurements (1) it is possible to write

$$x_3\,(k+1)\,\mathbf{d}\,(k+1) = \mathbf{s}\,(k+1) - \boldsymbol{x}_1\,(k+1)\,. \qquad (30)$$

In a similar way to the development of Section III, left multiply both sides of (30) by $\mathbf{d}^T\,(k+1)$ and use (2), which yields

$$x_3(k+1) = \mathbf{d}\,(k+1)\cdot\mathbf{s}\,(k+1) - \mathbf{d}\,(k+1)\cdot\boldsymbol{x}_1\,(k) - T\mathbf{d}\,(k+1)\cdot\boldsymbol{x}_2\,(k) - \mathbf{d}\,(k+1)\cdot\mathbf{u}\,(k)\,. \qquad (31)$$

Once again, the evolution of $x_3\,(k)$ described by (31) is undesirable as $x_3\,(k+1)$ does not depend on $x_3\,(k)$. In order to avoid that, add and subtract $\mathbf{d}\,(k+1)\cdot\mathbf{s}\,(k)$ to the right side of (31) and notice that

$$\mathbf{d}\,(k+1)\cdot\mathbf{s}\,(k) - \mathbf{d}\,(k+1)\cdot\boldsymbol{x}_1\,(k) = \mathbf{d}\,(k+1)\cdot\mathbf{d}\,(k)\,x_3\,(k)\,,$$

which allows to rewrite (31) as

$$x_3(k+1) = -T\mathbf{d}\,(k+1)\cdot\boldsymbol{x}_2\,(k) + \mathbf{d}\,(k+1)\cdot\mathbf{d}\,(k)\,x_3\,(k) - \mathbf{d}\,(k+1)\cdot\mathbf{u}\,(k) + \mathbf{d}\,(k+1)\cdot\left[\mathbf{s}\,(k+1) - \mathbf{s}\,(k)\right].$$

Define the augmented state vector

$$\boldsymbol{x}\,(k) := \begin{bmatrix} \boldsymbol{x}_1\,(k) \\ \boldsymbol{x}_2\,(k) \\ x_3\,(k) \end{bmatrix} \in \mathbb{R}^{3+3+1}.$$

Discarding the original nonlinear output (1) and considering (30) instead allows to write the discrete-time linear system

$$\left\{\begin{array}{l} \boldsymbol{x}\,(k+1) = \boldsymbol{\mathcal{A}}\,(k)\,\boldsymbol{x}\,(k) + \boldsymbol{\mathcal{B}}\,(k)\,\boldsymbol{u}\,(k) \\ \boldsymbol{y}\,(k+1) = \boldsymbol{\mathcal{C}}\,(k+1)\,\boldsymbol{x}\,(k+1) \end{array}\right. , \qquad (32)$$

where

$$\boldsymbol{\mathcal{A}}\,(k) := \begin{bmatrix} \mathbf{I} & T\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -T\mathbf{d}^T\,(k+1) & \mathbf{d}\,(k+1)\cdot\mathbf{d}\,(k) \end{bmatrix} \in \mathbb{R}^{7\times 7},$$

$$\boldsymbol{\mathcal{B}}\,(k) := \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ -\mathbf{d}^T\,(k+1) & \mathbf{d}^T\,(k+1) \end{bmatrix} \in \mathbb{R}^{7\times 6},$$

$$\boldsymbol{\mathcal{C}}\,(k) := \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{d}\,(k) \end{bmatrix} \in \mathbb{R}^{3\times 7},$$

and

$$\boldsymbol{u}\,(k) = \begin{bmatrix} \mathbf{u}\,(k) \\ \mathbf{s}\,(k+1) - \mathbf{s}\,(k) \end{bmatrix} \in \mathbb{R}^{3+3}.$$

In order to characterize the observability of the LTV system (32), consider the Lyapunov state transformation

$$\mathbf{z}(k) = \operatorname{diag}\left(\mathbf{I}, \mathbf{I}, -1\right)\boldsymbol{x}(\mathrm{k}).$$

The new system dynamics read as

$$\left\{\begin{array}{l} \mathbf{z}\,(k+1) = \mathbf{A}(k)\mathbf{z}(k) + \operatorname{diag}\left(\mathbf{I}, \mathbf{I}, -1\right)\boldsymbol{\mathcal{B}}(\mathrm{k})\boldsymbol{u}(\mathrm{k}) \\ \boldsymbol{y}\,(k+1) = \mathbf{C}\,(k+1)\,\mathbf{z}\,(k+1) \end{array}\right. .$$

Notice that the new system matrices $\mathbf{A}(t)$ and $\mathbf{C}(t)$ are those of the LTV system (10). This immediately allows to characterize the observability of the LTV system (32) with the following theorem, as both systems are related by a Lyapunov transformation [15].

Theorem 3: Under Assumption 1, the discrete-time linear system (32) is observable on $[k, k+3]$, for a fixed $k \geq k_0$, if and only if (11) holds.

As in Section III-B, the relation between the augmented system (32) and the original nonlinear system (4) is established in the following theorem.

Theorem 4: Consider Assumption 1 and suppose that (11) holds. Then:

i) the initial condition of (32) corresponds to the initial condition (4), i.e.,

$$\left\{\begin{array}{l} \mathbf{x}_1\,(k_0) = \mathbf{p}\,(k_0) \\ \mathbf{x}_2\,(k_0) = \mathbf{v}_f\,(k_0) \\ x_3\,(k_0) = \left\|\mathbf{s}\,(k_0) - \mathbf{p}\,(k_0)\right\| \end{array}\right. ;$$

ii) the nonlinear system (4) is observable in the sense that, given the system inputs $\mathbf{u}(k)$ and $\mathbf{s}\,(k+1) - \mathbf{s}\,(k)$ and the system output $\mathbf{y}(k)$, for $k = k_0$, k_0+1, $k = k_0+2$, its initial condition is uniquely determined; and

iii) an observer for the linear system (32) with globally exponentially stable error dynamics is also an observer for the nonlinear system (4), with globally exponentially stable error dynamics.

The proof follows similar steps to that of Theorem 2 and therefore it is omitted.

V. Simulation results

A numerical simulation is presented in this section in order to demonstrate the achievable estimation performance with the proposed solutions. Due to space limitations, only the problem of source localization is addressed here. Moreover, these are only preliminary results: extensive Monte Carlo simulations and comparison with the Extended Kalman filter will be performed in the future.

The initial position of the source is $\mathbf{s}(0) = [5\,0\,0]^T$ m, while the agent starts in the origin. The drift velocity of the source is $\mathbf{v}_s(t) = [1\,0\,0]^T$ m/s and the fluid velocity is $\mathbf{v}_f(t) = [-0.5\,0\,0]^T$ m/s, meaning an overall relative drift velocity of $\mathbf{v}_{sa}(t) = [1.5\,0\,0]^T$ m/s. The trajectories of the agent and the source are depicted in Fig. 1. Notice the rich trajectory described by the agent, ensuring that uniform complete observability is attained.

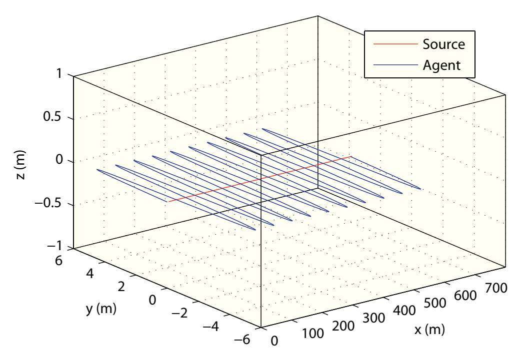
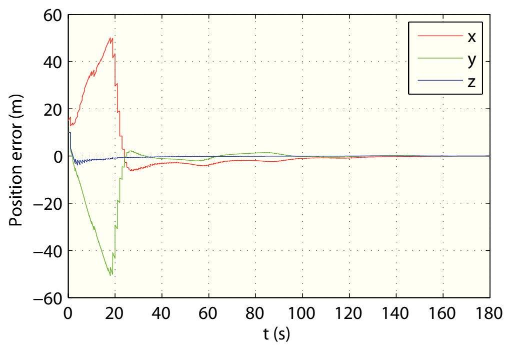


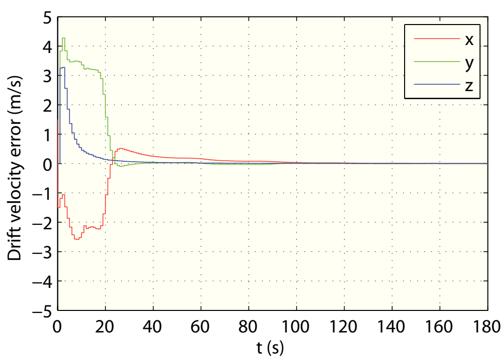
Fig. 1. Trajectories described by the agent and the source

A sampling period of $T = 1$ s was employed for the bearing measurements, which were rotated about random vectors of an angle that follows a zero-mean white Gaussian noise distribution, with standard deviation of 1°. The agent relative velocity is assumed available at 100 Hz and it is corrupted by additive zero-mean white Gaussian noise, with standard deviation of $0.01\,\mathrm{m/s}$. Euler integration was employed for open-loop propagation of the relative position of the source between direction measurements. To tune the Kalman filter, the state disturbance covariance matrix was chosen as $\mathrm{diag}\left(10^{-2}\mathbf{I},\, 10^{-5}\mathbf{I},\, 10^{-2},\right)$ and the output noise variance was set to the identity. The initial condition is $[-10\;\; -10\;\; -10]^T$ m for the relative position and zero for the remaining states.

The initial convergence of the position and velocity errors is depicted in Fig. 2, whereas the initial evolution of the range errors is shown in Fig. 3. As it can be seen from the various plots, the convergence rate of the filter is quite high. Although it is not shown here due to space limitations, the



(a) Position error



(b) Drift velocity error

Fig. 2. Initial convergence of the estimation error

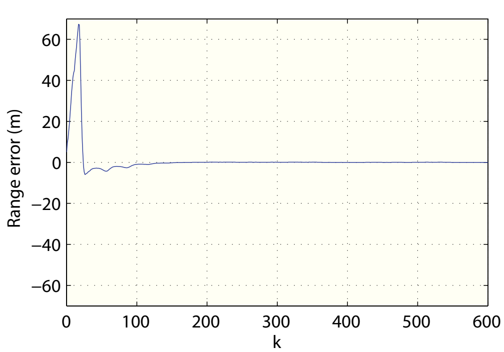


Fig. 3. Evolution of the estimation error of the range

steady-state error of the position error remains below 0.1 m for most of the time, while the velocity error remains below 0.002 m/s.

VI. Conclusions

This paper addressed the problems of source localization and navigation based on single discrete-time direction measurements, considering also constant unknown drift velocities. Based on previous work by the authors, discrete-time augmented linear systems were derived that were shown to mimic the evolution of the original nonlinear systems under appropriate observability conditions, with no conservativeness whatsoever. The Kalman filter provides the estimation solution, with globally exponentially stable error dynamics, and simulation results evidence both fast convergence and good performance in the presence of sensor noise. Future work will cover the comparison with existing techniques, in particular with the Extended Kalman Filter (EKF), which does not offer global convergence guarantees, and experimental validation.

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