Systems with Exponential Eigenfunctions and Exponential-Input/Constant-Output Operators

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***Abstract*—Recently, Vaidyanathan [1] and Ferreira [2] reported on nonlinear systems with exponential eigenfunctions. In this letter, we introduce a larger class of such systems, generated from a special family of exponential-input/constant-output operators that capture exponential signals from general inputs. In particular, we extend the classes of systems with exponential eigenfunctions defined in [1], [2] to include causal nonhomogeneous time-varying systems.**

***Index Terms*—Causality, eigenfunctions, exponential-input/constant-output operators, exponentials, homogeneous time-invariant systems, nonhomogeneous systems, nonlinear systems.**

I. Introduction

THE WELL-known exponential eigenfunction (EE) property of linear time-invariant (LTI) systems extends to homogeneous time-invariant (HTI) systems, as shown in a recent letter [1]. Additionally, that letter raised the challenge of finding larger classes of systems that respond with exponential outputs to exponential inputs. Subsequently, it was pointed out in [2] that a weighted sum of systems with the EE property has also the EE property. In [2], the weights are represented by functionals, i.e., signal-dependent gains, leading to examples of nonhomogeneous and time-varying systems with the EE property. In this letter, we introduce a new class of systems with the EE property, generated from exponential-input/constant-output (EICO) operators. We show that this new class extends those proposed in [1], [2].

To motivate the use of EICO operators, we emphasize that any system generated according to the method proposed in [2] is, in general, not causal. We overcome this limitation by replacing the aforementioned signal-dependent functionals by convenient operators (mappings between signals). These operators produce constant outputs in response to exponential inputs and arbitrary outputs to nonexponential inputs. We will see that a particular structure of EICO operators that capture exponential signals from general inputs, leads to a large variety of systems with the EE property, including causal nonhomogeneous time-varying ones.

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II. Nonlinear Systems with Exponential Eigenfunctions

Reference [1] proves that HTI systems have the EE property and presents the following example of such a system that is not LTI:

$$y(n) = \begin{cases} \dfrac{x^2(n)}{x(n-1)}, & \text{if } x(n-1) \neq 0 \\ 0, & \text{if } x(n-1) = 0. \end{cases} \tag{1}$$

In fact, the system (1) is nonlinear, and its response to an exponential input $x(n) = a^n$ is the same exponential multiplied by a complex constant $y(n) = aa^n$, i.e., it has the EE property.

More recently, reference [2] proposed a larger class of systems S with the EE property defined by

$$S[x] = \sum_{i \in J} F_i[x] H_i x \tag{2}$$

where H_i represents an HTI system, and F_i is a functional that maps the input signal into a complex value. We use a notation suggested by [2]: the arguments of nonlinear functionals or systems are written inside square brackets, but the brackets are dropped if the mapping is homogeneous. The rationale behind expression (2) is that a system does not lose the EE property when its output is multiplied by an input-dependent gain. These gains (i.e., the functionals $\{F_i\}$) induce possible nonhomogeneous and/or time-varying behaviors for the system S. As noted in [2], the systems $\{H_i\}$ need not to be HTI, they may be any systems with the EE property.

As an example of a nonhomogeneous and time-varying system that has the EE property, reference [2] takes the sum (2) to have only one term and specifies $F[x] = 1 + x(0)$. We observe that the resulting system is not causal because the output at any negative time instant depends on the future value of the input $x(0)$. In fact, all systems generated by (2) are not causal, except in two trivial cases. First, when the functionals $\{F_i\}$ do not depend on the input signal x, i.e., they are constants $F_i[x] = f_i$. Since any input-dependent functional F_i is a function of, at least, the value of the input at a specific time instant $x(n_0)$, the output $y(n)$ of the system S, for $n < n_0$, depends on a future value of the input. Second, when the signals are defined on a half-line $n \geq n_0$, and the functionals $\{F_i\}$ depend only on the value of the input signal at the initial time instant n_0, $F_i[x] = f_i(x(n_0))$.

III. Exponential-Input/Constant-Output Operators

In this section, we extend the class of systems with the EE property by replacing the functionals $\{F_i\}$ in (2) with operators

$\{T_i\}$ restricted to respond with constants to exponential inputs. Thus, we propose the class of systems S defined by

$$S[x] = \sum_{i \in J} T_i[x] H_i x \tag{3}$$

where H_i represents an HTI system. We call T_i an exponential-input/constant-output (EICO) operator because it maps exponential signals to complex constants

$$T_i[a^n] = t_i(a), \quad a \neq 0. \tag{4}$$

To show that the class of systems S given by (3) with T_i defined by (4) has the EE property, let $x(n) = a^n$, $a \neq 0$. From (4), $T_i[a^n] = t_i(a)$. Since H_i is homogeneous, $H_i a^n = h_i(a) a^n$, and the output of the system is the exponential input up to a complex scale factor $y(n) = (\sum_i t_i(a) h_i(a)) a^n$. When $a = 0$, $x(n) = 0$, $(H_i x)(n) = 0$, and $y(n) = 0$, independently of the operators $\{T_i\}$.

Obviously, the class of systems defined in (3) includes all the systems with the EE property proposed in [1], [2]. The relevant question is whether there are nonconstant operators that satisfy (4). The answer is affirmative. Next, we propose a simple way to generate such EICO operators. By using these operators, we design a wide class of systems with the EE property. Furthermore, unlike the systems generated according to (2), the proposed class includes time-varying, nonhomogeneous, causal systems. To construct such systems, we exploit an operator based on the nonlinear HTI system (1) proposed in [1], which has the capability of capturing exponential signals from general inputs.

The nonlinear system (1) is rewritten as $V[x]x$ according to the structure of (3), where $V[x]$ is the operator defined by

$$y = V[x] \quad \Leftrightarrow \quad y(n) = \begin{cases} \dfrac{x(n)}{x(n-1)}, & \text{if } x(n-1) \neq 0 \\ 0, & \text{if } x(n-1) = 0. \end{cases} \tag{5}$$

From this definition, we see that $V[a^n] = a$. Thus, V is an EICO operator, see (4), and in addition, it exhibits the useful property

$$V[V[a^n]] = 1, \quad a \neq 0. \tag{6}$$

According to (6), V is a particular operator that returns 1 when applied twice to any exponential signal a^n, $a \neq 0$.

Using the operator V, we define a broad class of EICO operators by

$$\tilde{T}[x] = (1 - V[V[x]]) G[x] + f(V[x](n)) \tag{7}$$

where

- $V[x](n)$ response of V to the input signal x, at time n;
- G operator;
- f function, i.e., a mapping between complex numbers (f can also be regarded as a time-invariant memoryless operator acting on $V[x]$).

Due to property (6), the first term of (7) is zero when the input is an exponential a^n, $a \neq 0$. As a consequence, $\tilde{T}[a^n] = f(a)$, $a \neq 0$, thus $\tilde{T}$ is an EICO operator for any choice of G and f. The complex value $f(a)$ is the gain associated to the exponential input a^n, $a \neq 0$. When the input x is not an exponential, it is possible to generate a wide class of output signals by specifying any mapping G in the first term of (7).

As an example of a system constructed with the operator $\tilde{T}$, we take the sum (3) to have only one term and the corresponding HTI system to be the identity operator. By choosing G in (7) to be the time-varying and causal operator $G[x] = n$, we obtain the following input–output relation for the system $\tilde{T}[x]x$:

$$y(n) = \begin{cases} \left(x(n) - \dfrac{x^2(n) x(n-2)}{x^2(n-1)} \right) n + \\ \quad f\left(\dfrac{x(n)}{x(n-1)} \right) x(n), & \text{if } x(n-1) \neq 0 \\ (n + f(0)) x(n), & \text{if } x(n-1) = 0. \end{cases} \tag{8}$$

This system has the EE property $[x(n) = a^n \Rightarrow y(n) = f(a) a^n]$. Furthermore, it is time-varying, nonhomogeneous, and the output at a given time instant does not depend on future values of the input, i.e., the system is causal.

IV. Remark

In the DSP community, an exponential signal is commonly defined by $x(n) = ca^n$, see [3]. Such signal is an eigenfunction of any system described by (3) when the EICO operators T_i take the form (7). In fact, the EICO operator given by (7) satisfies $\tilde{T}[ca^n] = f(a)$, $a \neq 0$, since, from (5), $V[ca^n] = V[a^n] = a$. As a final remark, we point out that, in opposition to homogeneous systems, nonhomogeneous ones without the EE property may still admit eigenfunctions a^n, $\forall a$, i.e., exponentials with $c = 1$ in the definition of [3]. An illustrative example follows.

Consider the system $Q[x] = L[x]x$ where the operator L is defined by

$$y = L[x] \quad \Leftrightarrow \quad y(n) = \begin{cases} \dfrac{\log x(n)}{n}, & \text{if } n \neq 0 \\ \log x(1), & \text{if } n = 0. \end{cases} \tag{9}$$

From (9), $L[a^n] = \log a$. Thus, the system Q has eigenfunctions a^n, $\forall a$

$$Q[a^n] = \log a \, a^n. \tag{10}$$

However, the response of Q to an exponential input signal that is exponential up to a scale factor $x(n) = ca^n$, $c \neq 1$ is not exponential. In fact, using (9), we obtain for $n \neq 0$

$$Q[ca^n] = \log a a^n + \log c \, \frac{a^n}{n}. \tag{11}$$

The previous example shows that some care must be taken when defining eigenfunctions of nonhomogeneous systems.

References

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