Distributed Optimization With Local Domains: Applications in MPC and Network Flows

João F. C. Mota, João M. F. Xavier, Pedro M. Q. Aguiar, and Markus Püschel

Abstract—We consider a network where each node has exclusive access to a local cost function. Our contribution is a communication-efficient distributed algorithm that finds a vector $x^*$ minimizing the sum of all the functions. We make the additional assumption that the functions have intersecting local domains, i.e., each function depends only on some components of the variable. Consequently, each node is interested in knowing only some components of $x^*$, not the entire vector. This allows improving communication-efficiency. We apply our algorithm to distributed model predictive control (D-MPC) and to network flow problems and show, through experiments on large networks, that the proposed algorithm requires less communications to converge than prior state-of-the-art algorithms.

Index Terms—Distributed algorithms, alternating direction method of multipliers (ADMM), model predictive control, network flows.

I. INTRODUCTION

Consider a network with $P$ nodes and the following problem:

$$\min_{x \in \mathbb{R}^n} f_1(x) + f_2(x) + \cdots + f_P(x),$$

where each function $f_p$ is known only at node $p$. We say an algorithm solving (1) is distributed if it uses no central node and no all-to-all communications. In a typical distributed algorithm, each node holds an estimate of a solution $x^*$ and iteratively updates and exchanges it with its neighbors. Such an algorithm implicitly assumes that each node is interested in knowing all the components of a solution $x^*$. While this holds for problems like consensus or distributed SVMs, there are important problems where it does not hold, especially in the context of large networks. Two examples we explore are distributed model predictive control (D-MPC) [1]–[3] and network flows [4].

We solve (1) assuming that each function $f_p$ depends only on a subset of the components of the variable $x \in \mathbb{R}^n$. This situation is illustrated in Fig. 1(a) with $n = 3$; there, for example, $f_1$ depends on $x_1$ and $x_2$, but not on $x_3$. To capture these dependencies, we write $x_S, S \subseteq \{1, \ldots, n\}$, to denote a subset of the components of $x$. For example, if $S = \{2, 4\}$, then $x_S = (x_2, x_4)$. With this notation, our goal is to solve

$$\min_{x \in \mathbb{R}^n} f_1(x_{S_1}) + f_2(x_{S_2}) + \cdots + f_P(x_{S_P}),$$

where $S_p$ is the set of components that the function $f_p$ depends on. Accordingly, node $p$ is interested in finding only the components of a solution $x^*$ that are indexed by $S_p$, i.e., $x^*_{S_p}$. This approach, however, introduces unnecessary communications, since nodes exchange more components than necessary. The goal of this paper is to design distributed algorithms solving (2) that use its structure to reduce the total number of communications.

Contributions. We propose a distributed algorithm that solves (2) in full generality, for arbitrary sets $S_p$. This is done by classifying its variable into two categories and designing algorithms for both. We then apply our algorithms to D-MPC and network flow problems. We show that, with our general solution, we can outperform prior algorithms, even application-specific ones. Due to space constraints, we omit some proofs and a detailed derivation for the non-connected case. Both, however, can be found in [5].

Related work. Many algorithms have been proposed for the global variable problem (1), including methods based on the alternating direction method of multipliers (ADMM) [6], [7]. As mentioned before, solving (2) with an algorithm designed for (1) introduces unnecessary communications.

To our knowledge, this is the first time that problem (2) has been explicitly stated in a distributed context. For example, [8, §7.2] proposes an algorithm for (2), but is not distributed in our sense, since it requires either a platform that supports all-to-all communications (in other words, a central node) or running consensus algorithms on each induced subgraph at each iteration [8, §10.1]. Thus, that algorithm is only distributed when every component induces subgraphs that are stars, in which case we say the variable is star-shaped. Actually, we found only one algorithm in the literature, namely [9], that is distributed (or that can easily be made distributed) for all the scenarios considered in this paper. Yet, that algorithm was proposed for a problem with a star-shaped variable: power system state estimation (our algorithm can be applied to this problem as well). Our simulations show that the algorithm in [9] requires always more communications than the algorithm we propose. Although

![Diagram](image-url)
we found just one (communication-efficient) distributed algorithm solving (2), many other algorithms solve particular instances of it; see, e.g., [5], [10], [11]. For example, in network flow problems, each component of the variable is associated to an edge of the network which, as we will see, enables writing them as (2) with a star-shaped variable. In this case, not only [8, §7.2] becomes distributed, but also gradient/subgradient methods can be applied directly and yield distributed algorithms [12]. Distributed Model Predictive Control (D-MPC) [1]–[3] is another problem that has been addressed with algorithms solving (2), again in the special case of a star-shaped variable. Such algorithms include fast gradient [13] and ADMM-based [13] methods (which applies [8, §7.2]). Akin to [8, §7.2], these methods were designed for the special case of star-shaped variables and become inefficient when applied to more generic cases. In spite of its generality, the algorithm we propose requires less communications than previous algorithms that were designed specifically for D-MPC or for network flow problems.

II. TERMINOLOGY AND PROBLEM STATEMENT

We first define communication network and variable connectivity:

Communication network. A communication network is represented as an undirected graph \( G = (V, E) \), where \( V = \{1, \ldots, P \} \) is the set of nodes and \( E \subseteq V \times V \) is the set of edges. Two nodes communicate if there is an edge connecting them in \( G \). We assume:

Assumption 1. \( G \) is connected and its topology does not change over time; also, a coloring scheme \( C \) of \( G \) is available beforehand.

A coloring scheme \( C \) is a set of numbers, called colors, assigned to the nodes such that two neighbors never have the same color. Given its importance in TDMA, a widespread protocol for avoiding packet collisions, there is a large literature on coloring networks, as briefly overviewed in [5, §3.1]. Our algorithm integrates naturally with TDMA, since both use coloring as a synchronization scheme: nodes work sequentially according to their colors, and nodes with the same color work in parallel. The difference is that TDMA uses a more restrictive coloring, as nodes within two hops cannot have the same color. Note that packet collision is often ignored in the design of distributed algorithms, as confirmed by the ubiquitous assumption that all nodes can communicate simultaneously. We associate with each node \( p \) in the network a function \( f_p : \mathbb{R}^{n_p} \rightarrow \mathbb{R} \cup \{ +\infty \} \), where \( 1 \leq n_p \leq n \), and make the

Assumption 2. Each function \( f_p \) is closed, proper, and convex over \( \mathbb{R}^{n_p} \), and is known only at node \( p \). The neighbors of node \( p \) know the set of components \( S_p \) that \( f_p \) depends on.

Since we allow \( f_p \) to take infinite values, constraints can be imposed via indicator functions, i.e., functions that evaluate to \( +\infty \) when the constraints are not satisfied, and to \( 0 \) otherwise.

Variable connectivity. Although each function \( f_p \) is available only at node \( p \), each component of the variable \( x \) may be associated with several nodes. Let \( x_i \) be a given component. The subgraph induced by \( x_i \) is \( \bar{G}_i = (V_i, \bar{E}_i) \subseteq G \), where \( V_i \) is the set of nodes whose functions depend on \( x_i \), and an edge \( (i, j) \in \bar{E}_i \) belongs to \( \bar{E}_i \) both if \( i \) and \( j \) are in \( V_i \). For example, the subgraph induced by \( x_1 \) in Fig. 1(a) consists of \( V_1 = \{1, 2, 4, 6\} \) and \( \bar{E}_1 = \{(1, 2), (1, 6), (2, 6)\} \). We say \( x_1 \) is connected if its induced subgraph is connected, and non-connected otherwise. Likewise, a variable is connected if all its components are connected, and non-connected if it has at least one non-connected component. In Fig. 1(a), the variable is non-connected, because \( x_1 \) induces a non-connected subgraph.

Problem statement. Given a network and a set of functions satisfying Assumptions 1 and 2, we design a distributed, communication-efficient algorithm that solves (2), with either a connected or a non-connected variable. Recall that a distributed algorithm uses neither a central node nor all-to-all communications.

III. CONNECTED CASE

In this section we derive a distributed algorithm for (2) assuming a connected variable. The main idea is to manipulate (2) to make the Extended ADMM (E-ADMM) [14] applicable. Our algorithm generalizes [7], which proposed an algorithm for (1).

Problem manipulation. Let \( x_l \) be a given component and \( G_l = (V_l, E_l) \) be the respective induced subgraph, assumed connected. Since all nodes in \( V_l \) are interested in \( x_l \), we create copies of \( x_l \) in each of those nodes: \( \bar{x}_l \) is the copy at node \( p \), and \( \bar{x}_l(p) = \{ \bar{x}_l(p) \}_{p \in \bar{S}_l} \) denotes all copies at node \( p \). We then rewrite (2) as

\[
\min_{\{\bar{x}_l^{(t)}\}_{t=1}} f_1(x_1^{(t)}) + f_2(x_2^{(t)}) + \cdots + f_P(x_P^{(t)})
\]

subject to

\[
x_l^{(t)} = \bar{x}_l^{(t)}, \quad (i, j) \in E_l, \quad l = 1, \ldots, n,
\]

where the optimization variable \( \{\bar{x}_l^{(t)}\}_{t=1} \) is the set of all copies. We used \( \bar{x}_l := \{\bar{x}_l(p)\}_{p \in V_l} \) to denote all copies of \( x_l \), which are located only in the nodes of \( G_l \). The constraints in (3) enforce equality among the copies of the same component: if two neighboring nodes \( i \) and \( j \) depend on \( x_l \), then \( x_l(i) = x_l(j) \) appears in the constraints of (3). We assume that any edge in the communication network is represented as the ordered pair \( (i, j) \in E_l \), with \( i < j \). As such, there are no repeated equations in (3). Problems (2) and (3) are equivalent because each induced subgraph is connected. We observe that \( x_l(i) = x_l(j) \), \( (i, j) \in E_l \), can be written as \( A_l x_l = 0 \), where \( A_l \) is the transposed node-arc incidence matrix of the subgraph \( G_l \). The node-arc incidence matrix associates each edge of a graph with a column of a matrix. The column associated with the edge \((i, j)\) has 1 in the \( i\)th entry, \(-1\) in the \( j\)th entry, and 0 elsewhere. We next partition the optimization variable according to the coloring scheme: for each \( l = 1, \ldots, n \), \( \bar{x}_l = (\bar{x}_l^1, \ldots, \bar{x}_l^C) \), where \( \bar{x}_l^c = \{\bar{x}_l(p)\}_{p \in V_l \cap C_c} \) if \( V_l \cap C_c \neq \emptyset \), and \( \bar{x}_l^c = \emptyset \) if \( V_l \cap C_c = \emptyset \). Also, \( C_c \) is the set of nodes that have color \( c \). Thus, \( \bar{x}_l^c \) is the set of copies of \( x_l \) held by the nodes that have color \( c \). If no node with color \( c \) depends on \( x_l \), then \( \bar{x}_l^c \) is empty.

A similar notation for the columns of the matrix \( A_l \) enables writing \( A_l x_l^c = \bar{A}_l^c \bar{x}_l^c \), and thus (3) equivalently as

\[
\min_{x_1, \ldots, x_C} \sum_{p \in \bar{S}_l} f_p(x_p^{(t)}) + \cdots + \sum_{p \in \bar{S}_l} f_p(x_p^{(t)})
\]

subject to

\[
a_l^1 \bar{x}_l^1 + \cdots + a_l^C \bar{x}_l^C = 0,
\]

where \( a_l^c = \{x_l^c\}_{t=1}^n \), and \( \bar{A}_l^c = \text{diag}(\bar{A}_l^1, \bar{A}_l^2, \ldots, \bar{A}_l^C) \) is the diagonal concatenation of the matrices \( \bar{A}_l^1, \bar{A}_l^2, \ldots, \bar{A}_l^C \). The format of (4) is exactly the one to which E-ADMM applies, as explained next.

E-ADMM. The Extended ADMM (E-ADMM) is a natural generalization of the Alternating Direction Method of Multipliers (ADMM). Given a set of closed, convex functions \( g_1, \ldots, g_C \), and a set of full column rank matrices \( E_1, \ldots, E_C \), E-ADMM solves

\[
\min_{x_1, \ldots, x_C} g_1(x_1) + \cdots + g_C(x_C)
\]

subject to

\[
E_1 x_1 + \cdots + E_C x_C = 0.
\]

It consists of iterating on \( k \) the following equations:

\[
x_k^1 = \text{arg min}_{x_1} L_\rho(x_1, x_2, \ldots, x_k; \lambda^k)
\]

\[
\vdots
\]

\[
x_k^C = \text{arg min}_{x_C} L_\rho(x_1^{k+1}, x_2^{k+1}, \ldots, x_C^{k+1}, x_C; \lambda^k)
\]

\[
\lambda^{k+1} = \lambda^k + \rho \sum_{c=1}^C E_c x_c^{k+1},
\]

where \( \rho \) is a positive constant.
where $L_p(x; \lambda) = \sum_{C=1}^{C_{\text{max}}} (g_j(x_j) + \lambda^T E_j x_j) + \frac{\rho}{2} \|\sum_{C=1}^{C_{\text{max}}} E_j x_j\|^2$ is the augmented Lagrangian of (5), $\lambda$ is the dual variable, and $\rho > 0$.

The original ADMM is recovered whenever $C = 2$, i.e., when there are only two terms in the sums of (5). The following theorem gathers some known convergence results for (6)-(8).

Theorem 1 ([14]–[17]). For each $c = 1, \ldots, C$, let $g_c : \mathbb{R}^{n_c} \to \mathbb{R}$ be closed and convex over $\mathbb{R}^{n_c}$ and dom $g_c \neq \emptyset$. Let each $E_c$ be an $m \times n_c$ matrix. Assume (5) is solvable and that either 1) $C = 2$ and each $E_c$ has full column rank, or 2) $C \geq 2$ and each $g_c$ is strongly convex. Then, the sequence $\{x_1^c, x_2^c, \ldots, x_{C-1}^c, \lambda^c \}$ generated by (6)-(8) converges to a primal-dual solution of (5). Furthermore, if each function $g_c$ is strongly convex, differentiable, and its gradient is Lipschitz-continuous, then linear convergence holds whenever $C = 2$, or $C > 2$ and $\rho$ in (8) is replaced by a small constant.

It is believed that (6)-(8) converges even when $C > 2$, each $g_c$ is closed and convex (not necessarily strongly convex), each matrix $E_c$ has full column rank, and is insufficiently orthogonal to the other matrices [18]. Such belief is supported by empirical evidence [5], [14] but proving it remains an open problem.

Applying E-ADMM. The clear correspondence between (4) and (5) makes (6)-(8) directly applicable to (4). Associate a dual variable $\lambda^{ij}_l$ to each constraint $x^{(i)}_l = x^{(j)}_l$ in (3). Translating (8) component-wise, $\lambda^{ij}_l$ is updated, for a given $(i, j) \in \mathcal{E}$, as

$$
\lambda^{ij}_{l,k+1} = \lambda^{ij}_l + \rho (x^{(i),k+1}_l - x^{(j),k+1}_l),
$$

where $x^{(i),k+1}_l$ is the estimate of $x_l$ at node $l$ after iteration $k$. This estimate is obtained from the sequence (6)-(7), where we will focus our attention now. This sequence will yield the synchronization mentioned in Section II: nodes work sequentially according to their colors, with the same-colored nodes working in parallel. In fact, each problem in (6)-(8) corresponds to one color. Moreover, each node works sequentially according to their colors, with the same-colored nodes working in parallel. Each node computes the vector $v$ in step 4, solves the optimization problem in step 6, and then sends the new estimates of $x_l$ to the neighbors that also depend on $x_l$, for $l \in S_p$. We introduced new notation in step 4: $C(p)$ is the color of node $p$. The optimization problem in step 6 involves the private function of node $p$, $f_p$, to which a linear and a quadratic term is added. Finally, note that the update of the dual variables in step 10 is different from (9). In particular, all the $\lambda$s at node $p$ were condensed into a single dual variable $\gamma_p$, since the optimization problem (12) does not depend on the individual $\lambda_p$s, but only on $\gamma_p := \sum_{j \in N_p \cap V_l} g(j - p)\lambda^{ij}_l$. If we replace $\lambda^{ij}_l$ by $\lambda^{ij}_l + \rho (x^{(i),k+1}_l - x^{(j),k+1}_l)$ in the definition of the $\gamma_p$-variable, we obtain the update of step 10. The extra “sign” in the previous expression (w.r.t. (9)) is necessary to take into account the extension of the definition of the dual variable $\lambda^{ij}_l$ for $i > j$ (see the proof of Lemma 4.6 in [5]).

Convergence. Algorithm 1 results from the application of E-ADMM to problem (4). Consequently, the conclusions of Theorem 1 apply if we prove that (4) satisfies the conditions of that theorem.

**Lemma 1.** Each matrix $A^c$ in (4) has full column rank.

**Proof:** Let $c$ be any color. Since $A^c = \text{diag}(A^\text{a}_1, A^\text{a}_2, \ldots, A^\text{a}_n)$, we have to prove that each $A^\text{a}_i$ has full column rank. Fix $c$ and $i$. If we prove that $(A^\text{a}_i)^\top A^\text{a}_i$ has full rank, then $A^\text{a}_i$ has full column rank. Since $A^\text{a}_i = \left[ A^\text{a}_1 A^\text{a}_2 \cdots A^\text{a}_n \right]$, $(A^\text{a}_i)^\top A^\text{a}_i$ corresponds to the $i$th block in the diagonal of $A^\text{a}_1 A^\text{a}_2 \cdots A^\text{a}_n$. By assumption, $G_i$ is connected, implying that each node in $G_i$ has at least one neighbor also in $G_i$; hence, each entry in the diagonal of $(A^\text{a}_i)^\top A^\text{a}_i$ is greater than zero. The same happens to the entries in the diagonal of $(A^\text{a}_i)^\top A^\text{a}_i$, which is also a diagonal matrix. This is because $(A^\text{a}_i)^\top A^\text{a}_i$ corresponds to the Laplacian entries of nodes that have the same color, which are never neighbors. Thus, $(A^\text{a}_i)^\top A^\text{a}_i$ has full rank.}

The following result is a consequence of Theorem 1 and Lemma 1.

**Algorithm 1 Algorithm for a connected variable**

**Initialization:** for all $p \in \mathcal{V}$, $l \in S_p$, set $\gamma^{(p)}_l = x^{(p)}_l = 0$, $k = 1$

1: repeat
2: for $c = 1, \ldots, C$ do
3: for all $p \in C_c$ [in parallel] do
4: for all $l \in S_p$ do
5: $v_l^{(p),k} = \gamma^{(p),k}_l - \rho \sum_{j \in N_p \cap V_l \cap C_c} x^{(j),k+1}_l - \rho \sum_{j \in N_p \cap V_l \cap C_j \cap c} x^{(j),k}_l$
6: end for
7: $\bigcup_{c \in C} \sum_{l \in S_p} L_l^{(p),k+1}$ as the solution of
8: arg min $f_p(x^{(p)}_l) + \sum_{l \in S_p} v_l^{(p),k+1,2} + \rho \sum_{l \in S_p} D_{p,l} \left(x_l^{(p)}\right)^2$
9: for each component $l \in S_p$, send $x^{(p),k+1}_l$ to $N_p \cap V_l$
10: for $p \in \mathcal{V}$ and $l \in S_p$ [in parallel] do
11: $\gamma^{(p),k+1}_l = \gamma^{(p),k}_l + \rho \sum_{j \in N_p \cap V_l} (x^{(j),k+1}_l - x^{(j),k+1}_l)$
12: end for
13: $k \leftarrow k + 1$
14: until some stopping criterion is met

Preprint submitted to IEEE Transactions on Automatic Control. Received: October 26, 2014 16:19:13 PST

0018-9286 (c) 2013 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.
Algorithm 2 Preprocessing step for a non-connected variable

1: Set $S'_p = \emptyset$ for all $p \in \mathcal{V}_t$, and $V'_t = V_t$ for all $t = \{1, \ldots, n\}$
2: for all $l \in \{1, \ldots, n\}$ such that $x_l$ is non-connected do
3: Compute a Steiner tree $(T_l, F_l)$ with $V_l$ as required nodes
4: Set $V'_l = T_l$ and $S'_l := T_l \backslash V_l$ (Steiner nodes)
5: end for

Corollary 1. Let Assumptions 1 and 2 hold and let the variable be connected. Let also one of the following conditions hold:

1) the network is bipartite, i.e., $C = 2$, or
2) each $\sum_{p \in E} f_p(x_{sp})$ is strongly convex, $c = 1, \ldots, C$.

Then, the sequence $\{x^{(k)}_p\}_{k=1}^{\infty}$ at node $p$, produced by Algorithm 1, converges to $\lambda$-Lipschitz-continuous gradient, then linear convergence holds in case $\sum_{l=1}^{n}$. The resulting problem is equivalent to (2). Algorithm 1 (resp. D-ADMM) involves communicating $\sum_{p=1}^{P} |S'_p|$ (resp. $nP$) numbers. When the variable is not global, $\sum_{p=1}^{P} |S'_p| < nP$, and thus there is a clear per-iteration gain in solving (2) with Algorithm 1.

IV. NON-CONNECTED CASE

When the variable is non-connected, problems (2) and (3) are no longer equivalent and, therefore, the previous derivations do not apply. We propose a small trick to make these problems equivalent.

Let $x_l$ be a component whose induced subgraph $G_l = (V_l, E_l)$ is non-connected. Then, the constraint $x^{(i)}_l = x^{(j)}_l$, $(i, j) \in E_l$, in (3) fails to enforce equality on all the copies of $x_l$. We propose replacing $G_l$ with a connected subgraph $G'_l \supseteq G_l$, obtained by adding nodes and edges to $G_l$. Since we seek to minimize communications, we want to add a minimal number of edges. This is exactly the optimal Steiner tree problem. Let $G = (V, E)$ and let $\mathcal{R} \subseteq \mathcal{V}$ be a set of required nodes. A Steiner tree is any tree $(T, F) \subseteq G$ that contains the required nodes: $\mathcal{R} \subseteq T$. The set of nodes in the tree that are not required, $\mathcal{S} := \mathcal{T} \backslash \mathcal{R}$, are called Steiner nodes. In our case, the set of required nodes is $\mathcal{R} = V_l$ and an optimal Steiner tree has a minimal number of edges. Although computing optimal Steiner trees is NP-hard, many approximation algorithms are available [19], [20], even distributed ones [21]. Note that the solution returned by our algorithm is independent of the optimality of the Steiner tree.

We propose computing a (not necessarily optimal) Steiner tree for each induced subgraph $G_l$ that is non-connected, as in Algorithm 2. The Steiner tree for component $x_l$ is given by $(T_l, F_l)$, where $V_l \subseteq T_l$ are the required nodes. The algorithm also computes a set $S'_p$ containing the variables for which node $p$ is Steiner. If node $p$ is not Steiner for any variable, then $S'_p = \emptyset$. By defining new induced subgraphs as $G'_l = (V'_l, E'_l)$, with $V'_l = T_l$ and $E'_l := E_l \cup F_l$, we create copies of $x_l$ in all nodes in $V'_l$ and replace $E_l$ in the constraints of (3) with $E'_l$. The resulting problem is equivalent to (2). Algorithm 1 can then be applied with minor modifications: replace every instance of $G_l = (V_l, E_l)$ with $G'_l = (V'_l, E'_l)$, and every instance of $S_p$ with $S'_p \cup S_p$ (see [5], §3.4.3.2 for more details). Algorithm 2 can be centralized or distributed (using, for example, [21] to compute Steiner trees) and is meant to be executed once, before solving the problem (or a batch). It requires knowledge of the communication network $G$ and the sets $S_p$, but not necessarily of the specific functions $f_p$. This preprocessing step can also be applied to the algorithm in [9].

V. APPLICATIONS

We now describe how distributed model predictive control (D-MPC) and network flows can be written as (2) and thus solved with Algorithm 1.

D-MPC. In model predictive control (MPC), a system is described at each time instant $t$ by its state-space $x[t] \in \mathbb{R}^n$. This state evolves as $x[t + 1] = \Theta(x[t], u[t])$, where $u[t] \in \mathbb{R}^m$ is a control input and $\Theta : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear function that models the system dynamics at time $t$. Given a time-horizon $T$, MPC consists of computing the state at time $t = 0$, computing optimal states and inputs for the next $T$ time steps, applying $u[0]$, setting $t = 0$, and repeating the process. The step of computing optimal states and inputs typically requires solving

$$\min_{x, u} \Phi(x[T]) + \sum_{t=0}^{T-1} \Psi(x[t], u[t]), \quad t = 0, \ldots, T - 1 \quad (13)$$

subject to $x[0] = x^0$, $x[t + 1] = \Theta(x[t], u[t]), \quad t = 0, \ldots, T - 1$. The set of nodes whose state and/or input influences $\mathcal{R}$ for all $t \geq 0$ is denoted $\mathcal{R}$. We assume $\mathcal{R}$ is a subset of the neighbors of node $p$. In other words, two systems that influence each other may be unable to communicate directly. This is illustrated in Fig. 2(b) where, for example, the state/input of node 3 influences the state evolution of node 1 (dotted arrow), but there is no communication link (solid line) between them. Finally, we assume functions $\Phi$ and $\Psi$ in (13) can be decomposed, respectively, as $\Phi(x[T]) = \sum_{p=1}^{P} \Phi_p(x(t), u(t))_{t \in \mathcal{J}(p)}$ and $\Psi(x[t], u[t]) = \sum_{p=1}^{P} \Psi_p(x(t), u(t))_{t \in \mathcal{J}(p)}$, where $\Phi_p$ and $\Psi_p$ are both associated to node $p$. In sum, we solve

$$\min_{x, u} \sum_{p=1}^{P} \left[ \Phi_{p}(x(t), u(t))_{t \in \mathcal{J}(p)} + \sum_{t=0}^{T-1} \Psi_{p}(x(t), u(t))_{t \in \mathcal{J}(p)} \right], \quad t = 0, \ldots, T - 1, \quad p = 1, \ldots, P, \quad (14)$$

where $x^0$ is the initial measurement at node $p$. The variable in (14) is $(\bar{x}, \bar{u}) := \left( \{x(t)\}_{t=0}^{T}, \{u(t)\}_{t=0}^{T} \right)$, where $x(t) := \{x_{SP}(t)\}_{S \subseteq \mathcal{V}, p = 1}^{T}$, and $\bar{u}(t) := \{u_{SP}(t)\}_{S \subseteq \mathcal{V}, p = 1}^{T}$. Problem (14) can be written as (2) by making $f_p(\{x_{sp} \cup S_p\} : \{\bar{x}(t)\}_{t=0}^{T}) = \Phi_p(\{x_{SP}(t)\}_{t=0}^{T}) + \sum_{t=0}^{T-1} \Psi_{p}(\{x(t), u(t)\}_{t \in \mathcal{J}(p)}), \quad \bar{u} = \{u_{SP}(t)\}_{t=0}^{T}$.
This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2014.2365686, IEEE Transactions on Automatic Control

Limited circulation. For review only

is the indicator function of the set \( \Gamma_1^p \), and \( \Gamma_1^p \) := \{ \{ x_j, u_j \} : x_j[t + 1] = \Theta_j(x_j[t], u_j[t]) \} \)

We illustrate in Fig. 2(a) the case where \( \Omega_p \subseteq N_p \cup \{ p \} \), i.e., the state of node \( p \) is influenced by its own state/input and by the states/inputs of the systems it communicates directly with. This corresponds star-shaped variable where the center of the star is node \( p \), whose state is \( x_p \). Particular cases of this model have been considered, for example, in [1], whose solutions are heuristics, and in [13], whose solutions are optimization-based. The model we propose is more general, since it can handle scenarios where interacting nodes do not need to communicate directly, or even scenarios with a non-connected variable. Both cases are shown in Fig. 2(b). For example, the subgraph induced by \( \{ x_3, u_3 \} \) contains nodes \{ 1, 2, 3, 4 \} and is connected. (Connectivity refers always to the communication network, which is represented by solid lines in the plots.) Nodes 1 and 3, however, do not communicate directly. This is an example of an induced subgraph that is not a star. On the other hand, the subgraph induced by \( \{ x_2, u_2 \} \) contains nodes \{ 1, 2, 3, 5 \} and is not connected, implying a non-connected variable.

Network flows. A network flow problem is typically formulated on a network with arcs (directed edges), where an arc from node \( i \) to node \( j \) indicates a flow in that direction [4]. In the example given in Fig. 3, there can be a flow from node 1 to node 5, but not the opposite. Every arc \((i, j) \in A\) has associated a non-negative variable \( x_{ij} \) representing the amount of flow in that arc (from \( i \) to \( j \)), and a cost function \( \phi_{ij}(x_{ij}) \) that depends only on \( x_{ij} \). The goal is to minimize the sum of all costs, while satisfying conservation of flow. External flow can be injected or extracted from a node, making that node a source or a sink, respectively. We represent the network of flows with the network-arc incidence matrix \( B \), where the column associated to an arc from node \( i \) to node \( j \) has a \(-1\) in the \( i\)th entry, a \( 1 \) in the \( j\)th entry, and zeros elsewhere. We assume the components of the variable \( x \) and the columns of \( B \) are in lexicographic order: e.g., the variable in Fig. 3 is \( x = (x_{12}, x_{15}, x_{23}, x_{24}, x_{34}, x_{35}, x_{46}, x_{56}) \).

Conservation of flow is expressed as \( Bx = d \), where \( d \in \mathbb{R}^d \) is the vector of external inputs/outputs. The entries of \( d \) sum up to zero and \( d_p < 0 \) (resp. \( d_p > 0 \)) if node \( p \) is a source (resp. sink). When node \( p \) is neither a source nor a sink, \( d_p = 0 \). We solve

\[
\begin{align*}
\text{minimize}_{\xi} \quad & \sum_{(i,j) \in A} \phi_{ij}(x_{ij}) \\
\text{subject to} \quad & Bx = d, \quad x \geq 0,
\end{align*}
\]

which is written as (2) with \( f_p(x_{ij}) = \frac{1}{2} \sum_{(p,j) \in A} \phi_{jp}(x_{jp}) + x_j - x_p \), and \( x \geq 0 \). This indicator function enforces the conservation of flow at node \( p \) and it only involves the variables \( x_{ij} \) \( \forall (i, j) \in A \) and \( x_j \).

We assume the communication network \( G = (V, E) \) consists of the underlying undirected network. This means nodes \( i \) and \( j \) can exchange messages directly, i.e., \( (i, j) \in E \) for \( i < j \), if either \( (i, j) \in A \) or \( (j, i) \in A \). Therefore, in contrast with the flows, messages do not necessarily need to be exchanged satisfying the direction of the arcs. In fact, messages and flows might represent different physical quantities, e.g., electricity and water. In problem (15), the subgraph induced by \( x_{ij} \) contains only nodes \( i \) and \( j \) and an edge connecting them. This makes the variable in (15) star-shaped.

VI. EXPERIMENTAL RESULTS

Using the applications from the previous section, we now show some illustrative experimental results for a connected variable. A thorough description of the experimental setup and more results (including for non-connected variables), can be found in [5, §4.4]. All experiments simulate a distributed environment in a single computer.\(^1\)

Network flows. In network flows we considered (15) with \( \phi_{ij}(x_{ij}) = x_{ij}(c_{ij} - x_{ij}) + 1_{\pi_{ij} \subseteq x_{ij}}(x_{ij}) \), where \( c_{ij} \) is the capacity of the arc \((i, j) \in A\), and used a randomly generated Barabasi network with 2000 nodes and 3996 edges. The results are shown in Fig. 4, which shows the relative error \( \| x^k - x^\star \| / \| x^\star \| \) versus the number of communication steps. Here, \( x^k \) is the concatenation of the estimates at all nodes, and \( x^\star \) is the optimal solution, computed in a centralized way. One communication step (CS) consists of all nodes communicating their current estimates to their neighbors; the total number of CSs is thus proportional to the total number of communications. We compared the proposed algorithm (Alg. 1) with [8] and [9], whose algorithms coincide for this problem, and Nesterov’s fast gradient algorithm [22]. In fact, when the variable is star-shaped, any gradient algorithm can be applied and becomes distributed (since the center of the star can act as a central node for that component). Fig. 4 shows that our proposed algorithm requires

\(^1\)Code that reproduces these experiments can be found in http://www.ee.ucl.ac.uk/~jmota/_static/DADMMp_reproducible.zip.
the least number of CSs to achieve any relative error between $10^{-1}$ and $10^{-4}$.

**D-MPC.** The results for D-MPC are shown in Fig. 2, where we used two different scenarios: 1) the network has 4941 nodes and 6594 edges (topology of the Western States Power Grid [23]), the variable is star-shaped, and all subsystems are stable; 2) the network has 100 nodes and 196 edges (randomly generated Barabasi network), the variable is connected but not star-shaped, and most subsystems are unstable. See [5, §4.4] for a description on how we generated the system interactions in scenario 2).

In both scenarios, we considered (14) with linear, time-invariant $\Phi_{ij}$'s (system interactions) and quadratic $\Psi_{ij}$'s and $\Psi_{ij}$. The size of the state (resp. input) at each node was always $m_p = 3$ (resp. $m_p = 1$), and the time-horizon was $T = 5$. This means the optimization variable in (14) had dimensions 24705 in scenario 1) and dimensions 500 in scenario 2). Fig. 5(a) shows the results for scenario 1) and Fig. 5(b) shows the results for scenario 2). Note that the only prior distributed algorithm that can solve problems for generic connected variables is [9], which is why only Alg. 1 and [9] appear in Fig. 5(b). For a star-shaped variable (Fig. 5(a)), both [8] and gradient-based methods [22] yield distributed algorithms. In both scenarios, the proposed algorithm (Alg. 1) was the algorithm that performed the best, requiring uniformly less CSs to achieve any relative error between $10^{-1}$ and $10^{-4}$.

**VII. CONCLUSIONS**

We designed algorithms for distributed optimization problems where the function at each node depends on arbitrary components of the variable, rather than on all of them. We classify an optimization variable as connected or non-connected, and propose an algorithm for each. Our algorithms require a network coloring scheme, and their convergence is guaranteed only when the network is bipartite or when the objectives are strongly convex. In practice, however, the algorithms converge even when none of these conditions is met. Moreover, experimental results show that our algorithms require less communications than prior algorithms to solve D-MPC or network flow problems to arbitrary levels of accuracy.

**REFERENCES**