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Brief paper

# Robust global trajectory tracking for a class of underactuated vehicles[☆]

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ABSTRACT

In this paper, we tackle the problem of trajectory tracking for a particular class of underactuated vehicles with full torque actuation and a single force direction (thrust), which is fixed relative to a body attached frame. Additionally, we consider that thrust reversal is not available. Under some given assumptions, the control law that we propose is able to track a smooth reference position trajectory while minimizing the angular distance to a desired orientation. This objective is achieved robustly, with respect to bounded state disturbances, and globally, in the sense that it is achieved regardless of the initial state of the vehicle. The proposed controller is tested in an experimental setup, using a small scale quadrotor vehicle and a motion capture system.



## 1. Introduction

In recent years, the advent of miniaturized electronics allowed for the development of small-scale aerial vehicles that are able to perform efficiently a number of different tasks, such as surveillance, targeting, structure inspection, among others (cf. Herrick, 2000 and Kinsey, Eustice, & Whitcomb, 2006). In order to take full advantage of the capabilities of these vehicles, several controllers have been proposed that make use of different parametrizations of the attitude of the system, such as: Euler-angles, quaternions, rotation matrix and angle-axis parametrization, just to name a few. Euler-angles arise when linearization is used in the controller design and even though they are very intuitive in nature and might be used effectively for local stabilization around a given set-point, they are not singularity-free, i.e., there exist points in the attitude space that cannot be represented with a given set of Euler-angles, so they cannot be used for the purpose of global stabilization. The rotation matrix provides a singularity-free injective representation of the orientation of the vehicle, and it can be used for controller design as suggested by Koditschek (1989). The unit quaternions and the angle-axis are representations of attitude that provide a double cover of the attitude space, meaning that for every orientation there exist two quaternions (or two different angle-axis combinations) that represent that rotation. So, even though they are singularity-free, they might lead to *inconsistent* behavior, namely the unwinding phenomenon (cf. Mayhew, Sanfelice, & Teel, 2011a). In order to avoid such problems, one is required to select the sign of the unit quaternion so that the kinematic equations of motion are satisfied. In practice, a memory state is required to keep track of past values, as suggested by Mayhew, Sanfelice, and Teel (2013). For a more in-depth discussion on attitude representation, the reader is referred to the work of Shuster (1993).

Every attitude representation has its advantages and drawbacks and, depending on the particular application at hand, some might prove more useful than others. In particular, unit quaternion representations have been applied to the control of spacecraft by Joshi, Kelkar, and Wen (1995), Kristiansen, Nicklasson, and Gravdahl (2009), Li, Ding, and Li (2010), Wisniewski and Kulczycki (2003), unmanned aerial vehicles by Tayebi and McGilvray (2006)

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and underwater vehicles by Fjellstad and Fossen (1994). A good overview of these different control techniques can be found in the work by Chaturvedi, Sanyal, and McClamroch (2011).

Despite their relative success, the aforementioned papers provide a control solution for fully actuated vehicles, ruling out very common vehicles such as helicopters and underwater vehicles. In order to address this issue, other control solutions have been presented by Aguiar and Hespanha (2007), Goodarzi, Lee, and Lee (2013) and Lee, Leok, and Harris McClamroch (2011). However, such strategies rely on continuous controllers and it has been shown by Bhat and Bernstein (2000) that global asymptotic stabilization of a given set-point is not possible by means of continuous feedback. In order to solve this problem, discontinuous control laws have been proposed (see e.g. Fjellstad & Fossen, 1994) but these are not robust to small measurement noise, as shown by Mayhew and Teel (2011a). Recent advances in hybrid control theory have shown that hybrid systems satisfying the so-called *hybrid basic conditions* are inherently robust to small measurement noise (cf. Goebel, Sanfelice, & Teel, 2012), making hybrid control techniques a suitable candidate for the problem at hand. In fact, hybrid control strategies using both quaternion feedback and rotation matrix feedback have been proposed by Mayhew et al. (2011a) and Mayhew and Teel (2011b), respectively.

In this paper, we will make use of the hybrid quaternion feedback strategy that is presented by Mayhew et al. (2011a), in order to design a controller for a class of underactuated vehicles that have a single force direction, known as thrust, and full torque actuation. Resorting to the backstepping of hybrid feedback laws given by Mayhew, Sanfelice, and Teel (2011b), we design a controller that is able to globally asymptotically stabilize a given smooth reference position trajectory while minimizing the rotation angle to a given attitude configuration. The proposed strategy is, in part, similar to that of Zhao, Dong, and Farrell (2013), however, the controller we propose includes an integral term that makes it robust to static acceleration perturbations, we use a robust hybrid system in order to extract the desired unit quaternion and our solution is evaluated in an experimental setup using an optical motion capture system. A preliminary version of this article that does not consider the additive disturbance and without the experimental results was presented at the 2013 American Control Conference (cf. Casau, Sanfelice, Cunha, Cabecinhas, & Silvestre, 2013). Another preliminary version of the present work with a system model that does not include the attitude dynamics was presented at the 2014 International Conference on Robotics and Automation (cf. Casau, Sanfelice, Cunha, Cabecinhas, & Silvestre, 2014).

The remainder of the paper is organized as follows. In Section 2, we present some of the notation and basic concepts that are used throughout the paper. In Section 3, we rigorously define the problem at hand and present some of the assumptions that render the proposed controller a feasible solution to the given problem. In Section 4, we devise a controller for the position subsystem, considering the orientation and the thrust as inputs, while in Sections 5 and 6 we follow the backstepping procedures in order to devise a controller in terms of the torque and the thrust. In Section 7, we present some experimental results. Finally, Section 8 provides some concluding remarks to this work.

## 2. Preliminaries

$\mathbb{N}$ denotes the set of natural numbers; $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space equipped with the inner product $\langle x, y\rangle := x^\top y$ for each $x, y \in \mathbb{R}^n$ which induces the norm $|x| := \sqrt{\langle x, x\rangle}$; $\mathbb{R}^{m\times n}$ denotes the set of $m\times n$ matrices; $\mathrm{vec} : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn}$ is given by $\mathrm{vec}(A) := \left[e_1{}^\top A^\top \cdots e_n{}^\top A^\top\right]^\top$ for each $A \in \mathbb{R}^{m\times n}$, where $e_i \in \mathbb{R}^n$ is a vector of zeros except for the $i$th entry which is 1; $|v|_\infty := \max_{i\in\{1,\dots,n\}} v_i$ for each $v \in \mathbb{R}^n$; $|A|_2$ denotes the maximum singular value of a matrix $A \in \mathbb{R}^{m\times n}$; given $M > 0$, we have that $M\mathbb{B} := \{x \in \mathbb{R}^n : |x| \le M\}$; given a set valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, the range of $M$ is the set $\mathrm{rge}M = \{y \in \mathbb{R}^n : \exists x \in \mathbb{R}^m \text{such that } y \in M(x)\}$.

We follow the same notation of Magnus and Neudecker (1985) to represent the derivatives of differentiable functions. Let $F : \mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q}$ be a differentiable function, then

$$\mathcal{D}_X (F) := \frac{\partial \mathrm{vec}(F)}{\partial \mathrm{vec}(X)^\top}. \tag{1}$$

We also define the saturation function:

**Definition 1.** A *K-saturation function* is a smooth strictly increasing function $\sigma_K : \mathbb{R} \to \mathbb{R}$ that satisfies the following properties: (1) $\sigma_K(0) = 0$, (2) $s\sigma_K(s) > 0$ for all $s \neq 0$, (3) $\lim_{s\to\pm\infty} \sigma_K(s) = \pm K$, for some $K > 0$. Moreover, for each $x \in \mathbb{R}^n$ we define

$$\Sigma_K(x) := \begin{bmatrix}\sigma_K(x_1) & \cdots & \sigma_K(x_n)\end{bmatrix}^\top. \quad \square$$

The attitude of a rigid-body can be described by an element $R$ of $SO(3)$ given by $SO(3) := \{R \in \mathbb{R}^{3\times 3} : R^\top R = I_3, \det(R) = 1\}$. Flows in $SO(3)$ satisfy $\dot{R} = RS(\omega)$, for some $\omega \in \mathbb{R}^3$, and $S(\omega)$ is such that $S(\omega)v = \omega \times v$ for each $\omega, v \in \mathbb{R}^3$ (cf. Bullo & Lewis, 2005, Section 4.1.5). Let $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ denote the $n$-dimensional sphere, defined by $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : x^\top x = 1\}$. The attitude of a rigid body may also be represented by unit quaternions $q := [\eta\ \epsilon^\top]^\top := (\eta, \epsilon)$, where $\eta$ and $\epsilon$ denote the scalar and vector components of $q \in \mathbb{S}^3$, respectively. The mapping $R : \mathbb{S}^3 \to SO(3)$, given by

$$R(q) := I_3 + 2\eta S(\epsilon) + 2S(\epsilon)^2, \tag{2}$$

maps a given unit quaternion to a rotation matrix (cf. Wertz, 1978, Eqs. (12)–(47)). This map is a double cover of $SO(3)$, since $R(q) = R(-q)$. It is important to note that for any continuous path $\mathcal{R} : [0, 1] \mapsto SO(3)$ and for any $q(0) \in \mathbb{S}^3$ such that $R(q(0)) = \mathcal{R}(0)$, there exists a unique continuous path $q : [0, 1] \mapsto \mathbb{S}^3$ such that $R(q(t)) = \mathcal{R}(t)$ for all $t \in [0, 1]$ (cf. Bhat & Bernstein, 2000). This is known as the *path lifting property* and, in particular, it means that the solution $t \mapsto R(t)$ to $\dot{R} = RS(\omega)$ can be uniquely lifted to a path $t \mapsto q(t)$ in $\mathbb{S}^3$ that satisfies

$$\dot{q} = \frac{1}{2}\begin{bmatrix}-\epsilon^\top \\ \eta I_3 + S(\epsilon)\end{bmatrix}\omega := \frac{1}{2}\Pi(q)\omega.$$

We make use of recent developments on hybrid systems theory in Goebel et al. (2012). Under this framework, a hybrid system $\mathcal{H}$ is defined as

$$\mathcal{H} = \begin{cases}\dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D,\end{cases} \tag{3}$$

where the data $(C, F, D, G)$ is given as follows: the set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the *flow map* and governs the continuous dynamics (also known as flows) of the hybrid system; the set $C \subset \mathbb{R}^n$ is the *flow set* and defines the set of points where the system is allowed to flow; the set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the *jump map* and defines the behavior of the system during jumps; the set $D \subset \mathbb{R}^n$ is the *jump set* and defines the set of points where the system is allowed to jump. A solution $x$ to $\mathcal{H}$ is parametrized by $(t, j)$, where $t$ denotes ordinary time and $j$ denotes the jump time, and its domain $\mathrm{dom}\, x \subset \mathbb{R}_{\ge 0} \times \mathbb{N}$ is a hybrid time domain: for each $(T, J) \in \mathrm{dom}\, x$, $\mathrm{dom}\, x \cap ([0, T] \times \{0, 1, \dots, J\})$ can be written in the form $\cup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \le t_1 \le t_2 \le \cdots \le t_J$, where $t_j$'s define the jump times. For a definition of asymptotic stability for hybrid systems see Goebel et al. (2012, Definition 7.1).

## 3. Problem formulation

In this paper, we consider the problem of designing a controller for a class of rigid bodies with a single thrust direction and full torque actuation. This includes, for example, different types of helicopter vehicles. For controller design purposes, we consider that the dynamics of such vehicles can be described by the following set of differential equations:

$$\dot{p} = v, \qquad \dot{v} = -Re_3\frac{T}{m} + ge_3 + L(p, v)b, \tag{4a}$$

$$\dot{R} = RS(\omega), \qquad \dot{\omega} = -J^{-1}S(\omega)J\omega + J^{-1}M, \tag{4b}$$

where $p \in \mathbb{R}^3$ denotes the position of the rigid body in the inertial reference frame, $v \in \mathbb{R}^3$ represents its linear velocity, expressed in inertial coordinates, $R \in SO(3)$ represents the orientation of the body fixed frame with respect to the inertial reference frame, $\omega \in \mathbb{R}^3$ denotes the angular velocity, expressed in the body attached frame, $g \in \mathbb{R}$ denotes the acceleration of gravity, $(p, v) \mapsto L(p, v) \in \mathbb{R}^{3\times\ell}$ is smooth function that represents state dependent disturbances that scale linearly with an unknown constant parameter $b \in \mathbb{R}^\ell$ for some $\ell \in \mathbb{N}$, $T \in \mathbb{R}$ is the thrust magnitude, $M \in \mathbb{R}^3$ is the torque, $m \in \mathbb{R}$ denotes the mass of the rigid body and $J \in \mathbb{R}^{3\times3}$ denotes its tensor of inertia. This model is similar to those that were used by Frazzoli, Dahleh, and Feron (2000) and Hua, Hamel, Morin, and Samson (2009). For more details on aircraft models, the reader is referred to Betty (1986), Padfield (2007) and references therein.

Suppose that we are given a function $t \mapsto (p_d^{(4)}(t), \dot{\omega}_d(t)) \in M_p\mathbb{B} \times M_\omega\mathbb{B}$ for some $M_p, M_\omega > 0$ and for each $t \geq 0$. Then, the position and attitude reference trajectories, denoted by $p_d$ and $R_d$, respectively, are obtained by integration of this function, given a set of suitable initial conditions. In particular, the attitude trajectory $t \mapsto R_d(t)$ is obtained by the integration of the differential equation $\dot{R}_d = R_dS(\omega_d)$, hence guaranteeing that $R_d(t)$ belongs to $SO(3)$ for each $t \geq 0$.

This procedure gives rise to a map $t \mapsto r(t) \in \mathbb{R}^{12} \times SO(3) \times \mathbb{R}^3$ defined for each $t \geq 0$, where

$$r(t) := (p_d(t), p_d^{(1)}(t), p_d^{(2)}(t), p_d^{(3)}(t), R_d(t), \omega_d(t)) \tag{5}$$

collects not only the position and attitude trajectories, but their derivatives up to a certain order. In the sequel, we restrict our attention to bounded reference trajectories and disturbances satisfying the following assumption.

**Assumption 1.** Given $M_p, M_\omega > 0$, a *reference trajectory* is a solution $r$ to

$$\dot{r} \in F_d(r) := \left(p_d^{(1)}, p_d^{(2)}, p_d^{(3)}, M_p\mathbb{B}, R_dS(\omega_d), M_\omega\mathbb{B}\right), \tag{6}$$

such that rger $\in \Omega_r$ for some compact set $\Omega_r \subset \mathbb{R}^{12} \times SO(3) \times \mathbb{R}^3$, satisfying $e_3{}^\top R_d(t)e_3 \geq 0$ for each $t \geq 0$. Moreover, for each disturbance $(p, v, b) \mapsto L(p, v)b$ for (4), the following holds:

$$\sup_{r\in\Omega_r} \left|p_d^{(2)}\right| + \left(\sqrt{3} + \sqrt{\ell} \sup_{(p,v)\in\mathbb{R}^6} |L(p, v)|_2\right) |b|_\infty < g. \tag{7}$$

Notice that the model disturbances $(p, v, b) \mapsto L(p, v)b$ must be bounded. If the disturbance term does not satisfy this requirement, then the results presented in this paper do not hold globally, but rather on a subset of the state space where (7) is satisfied. Additionally, (7) is required to guarantee that the thrust is positive, as described in the problem statement below. Moreover, it is not possible for an underactuated vehicle to track an arbitrary reference trajectory (cf. Levine & Müllhaupt, 2011). Therefore, given a reference trajectory $r$ satisfying Assumption 1, the controller proposed in this paper is able to track the attitude trajectory $R_0(r, \Upsilon)$ obtained by solving the optimization problem

$$\begin{aligned} &\text{minimize} \quad \frac{1}{2}\text{trace}\left(I_3 - RR_d{}^\top\right) \\ &\text{subject to} \quad R \in \Upsilon \end{aligned} \tag{8}$$

where $\Upsilon \subset SO(3)$ is to be defined during the design of the controller.

**Problem 1.** Design a hybrid controller

$$\dot{x}_c \in F_c(x) \quad x \in C_c, \qquad x_c^+ \in G_c(x) \quad x \in D_c, \tag{9}$$

with output $(T(x), M(x))$, where $x := (r, p, v, R, \omega, x_c)$ belongs to $\mathcal{X} := \Omega_r \times \mathbb{R}^3 \times \mathbb{R}^3 \times SO(3) \times \mathbb{R}^3 \times \mathcal{X}_c$, for some $\mathcal{X}_c$, such that the set

$$\mathcal{A} := \left\{x \in \mathcal{X} : p = p_d,\ v = p_d^{(1)},\ R = R_0(r, \Upsilon)\right\}, \tag{10}$$

is *globally asymptotically stable* for the closed-loop system resulting from the interconnection between (4) and the controller (9), and there exists $\overline{T} > 0$ such that $0 < T(x(t, j)) \leq \overline{T}$ for each $(t, j) \in \operatorname{dom} x$ and for each solution $x$ to the closed-loop system. □

To solve this problem, we separate it into three simpler tasks. In Section 4, we design a controller for the position subsystem and then, in Sections 5 and 6, we design a control law for the whole system using backstepping techniques.

## 4. Robust position tracking by saturated feedback

In this section, let us consider $R \in SO(3)$ as a virtual input. Then, given a reference trajectory satisfying Assumption 1, the position and velocity tracking errors are given by $p_0 := p - p_d$ and $v_0 := v - \dot{p}_d$, respectively. Then, using (4a), we find that the dynamics of the tracking errors are given by

$$\dot{p}_0 = v_0, \tag{11a}$$

$$\dot{v}_0 = -Re_3\frac{T}{m} + ge_3 + L(p, v)b - p_d^{(2)}. \tag{11b}$$

Since we are considering both $R \in SO(3)$ and $T \in \mathbb{R}$ as inputs, the term $-Re_3T/m$ can be set to an arbitrary vector $\mu \in \mathbb{R}^3 \setminus \{0\}$ using the thrust input $T(\mu) := m|\mu|$, and the attitude input as the solution to the optimization problem (8) with $\Upsilon_\mu := \{R \in SO(3) : Re_3 = -\mu/|\mu|\}$, as the constraint for $R$ which is given by

$$R_0(r, \Upsilon_\mu) = \left(I_3 + S(\gamma) + \frac{1}{1 - e_3{}^\top R_d{}^\top \frac{\mu}{|\mu|}} S(\gamma)^2\right) R_d, \tag{12}$$

where $\gamma := -S(R_de_3)\frac{\mu}{|\mu|}$, for each $\mu \in \mathbb{R}^3$ (cf. Frazzoli et al., 2000). Then, let us define the feedback law

$$\begin{aligned} \mu(r, p_0, v_0, z) := &-\Sigma_K(k_pp_0 + k_vv_0) - L(p, v)\,\Sigma_K(z) \\ &- ge_3 + p_d^{(2)}, \end{aligned} \tag{13}$$

where $\Sigma_K : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is a $K$-saturation function with the properties given in Definition 1, $k_p, k_v > 0$ and $z \in \mathbb{R}^\ell$ is an integral state satisfying $\dot{z} := k_zL(p, v)^\top \mathcal{D}_{v_0}\left(\overline{V}_0(p_0, v_0)\right)^\top$, for each $(r, p_0, v_0) \in \Omega_r \times \mathbb{R}^3 \times \mathbb{R}^3$, where

$$\overline{V}_0 := \sum_{i=1}^{3} \frac{1}{2}\begin{bmatrix}\sigma_K(\widetilde{r}_i) & e_i{}^\top v_0\end{bmatrix} P \begin{bmatrix}\sigma_K(\widetilde{r}_i) \\ e_i{}^\top v_0\end{bmatrix} + \int_0^{r_i} \sigma_K(\xi)\, d\xi, \tag{14}$$

with $\widetilde{r} := k_pp_0 + k_vv_0$, $\sigma_K$ given in Definition 1 and

$$P := \begin{bmatrix} \frac{k_v}{k_p}\beta & -\beta \\ -\beta & k_p \end{bmatrix},$$

for some $\beta \in (0, k_v)$.

Clearly, $R_0(r, \Upsilon_\mu)$ is not defined when $e_3{}^\top R_d\mu = |\mu|$. However, if $e_3{}^\top R_d(t)e_3 \geq 0$ for each $t \geq 0$ and (7) is satisfied then this situation does not happen for any solution to the closed-loop system. Replacing $(T, R) = (m|\mu|, R_0(r, \Upsilon_\mu))$ into (11), we obtain the closed-loop system

$$\begin{aligned}\dot{p}_0 &= v_0,\\ \dot{v}_0 &= -\Sigma_K(k_p p_0 + k_v v_0) + L(p, v)(b - \Sigma_K(z)), \qquad (15)\\ \dot{z} &= k_z L(p, v)^\top \mathcal{D}_{v_0}\left(\overline{V}_0(p_0, v_0)\right)^\top,\end{aligned}$$

with the important stability properties that are given in the following lemma whose proof is deferred to Appendix.

**Lemma 1.** *Let Assumption 1 hold. Then, there exists $K > |b|_\infty$ such that, for each $k_p, k_v, k_z > 0$, the set*

$$\mathcal{A}_0 := \{(p_0, v_0, z) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^\ell : p_0 = v_0 = 0\}, \qquad (16)$$

*is globally asymptotically stable for the system (15) and its solutions are bounded. Moreover, there exists $\overline{T} > 0$ such that $0 < T(\mu(r, p_0(t), v_0(t), z(t))) \leq \overline{T}$ for each $r \in \Omega_r$ and for each solution $t \mapsto (p_0(t), v_0(t), z(t))$, defined for each $t \geq 0$.*

The controller for the position subsystem presented in this section determines the acceleration of the vehicle and, as a consequence, it fixes a *desired* direction $R_0(r, \Upsilon_\mu)e_3$. The extra degree of freedom, consisting of rotations around $R_0(r, \Upsilon_\mu)e_3$, is used to minimize the distance to $R_d$. In this way, the user may specify attitude trajectories $R_d$ that are not feasible, but are mapped to the feasible trajectory $R_0(r, \Upsilon_\mu)$ in the sense that, for each $t \geq 0$, $R(t) = R_0(r(t), \Upsilon_{\mu(t)})$ and $p_0(t) = v_0(t) = 0$ satisfy (11), while $R(t) = R_d(t)$ may not.

## 5. Global asymptotic stabilization of the attitude kinematics by hybrid quaternion feedback

In this section, we develop a controller that solves Problem 1 when $\omega$ is taken as a virtual input. To do so, let us define the rotation error as $R_1 := RR_0{}^\top$, where $R_0$ is given by (12). Since $R_0(t) \equiv R_0(r(t), X_{\mu(t)})$ belongs to $SO(3)$ for each $t \geq 0$, then its derivative satisfies $\mathcal{D}_t(R_0(t)) = -\Gamma(R_0(t))R_0(t)\omega_0(t)$, for each $t \geq 0$, with

$$\Gamma(R) := -\begin{bmatrix} S(Re_1) & S(Re_2) & S(Re_3)\end{bmatrix}^\top.$$

Moreover, solving $\mathcal{D}_t(R_0(t))$ for $\omega_0$, we obtain

$$\omega_0 = -\frac{1}{2}R_0{}^\top\Gamma(R_0)^\top\mathcal{D}_t(R_0), \qquad (17)$$

and, from (4b), we conclude that $\dot{R}_1 = R_1 S(R_0(\omega - \omega_0))$.

The design of a controller such that $R = R_0$ is globally asymptotically stable is equivalent to the design of a controller that stabilizes $R_1 = I_3$. Although strategies for the global stabilization of an attitude reference by matrix feedback exist (cf. Mayhew & Teel, 2011b), it is not clear how they can be extended to the stabilization of the class of underactuated vehicles presented in this paper. Instead, we resort to attitude stabilization by hybrid quaternion feedback introduced in Mayhew et al. (2011a).

In this direction, let us point out that there exists a unique unit quaternion satisfying $R_1 = R(q_1)$ and the kinematic equations

$$\dot{q}_1 = \frac{1}{2}\Pi(q_1)R_0(\omega - \omega_0). \qquad (18)$$

In order to retrieve the unit quaternion uniquely, we make use of the robust path-lifting strategy that was introduced in Mayhew et al. (2013). We discuss the implementation of this technique at the end of Section 6, but, for now, we assume that $q_1$ is readily available from the measurements.

In standard backstepping we would add a feedforward term to $\omega$ in order to cancel out $\omega_0$. However, due to the presence of an unknown constant $b \in \mathbb{R}^\ell$ in the dynamics of the plant, we cannot determine $\omega_0$. Instead, we use an estimate $\omega_{0,1}$, given by $\omega_{0,1} := -\frac{1}{2}R_0{}^\top\Gamma(R_0)^\top\mathcal{D}_t(R_0)|_{b=b_1}$, which is in all aspects identical to (17) but where we replace the unknown disturbance $b$ by an estimate $b_1$. It is possible to verify that the difference between $\omega_{0,1}$ and $\omega_0$ is given by

$$\omega_{0,1} - \omega_0 = -\frac{1}{2}R_0{}^\top\Gamma(R_0)^\top\mathcal{D}_{v_0}(R_0)L(p, v)\widetilde{b}_1, \qquad (19)$$

where we have used the definition of the estimation error $\widetilde{b}_1 := b_1 - b$.

Let $\eta_1$ and $\epsilon_1$ denote the scalar and vector components of $q_1$, respectively, $H := \{-1, 1\}$, $\delta > 0$,

$$\begin{aligned}Q_\delta^+ &:= \{(q, h)\times \in \mathbb{S}^3 \times H : h\eta \geq -\delta\},\\ Q_\delta^- &:= \{(q, h)\times \in \mathbb{S}^3 \times H : h\eta \leq -\delta\},\end{aligned} \qquad (20)$$

$x_1 := (r, p_0, v_0, q_1, z, h, \widetilde{b}_1)$ belongs to $\mathcal{X}_1 := \Omega_r \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^3 \times \mathbb{R}^\ell \times H \times \mathbb{R}^\ell$ and

$$\dot{b}_1 := \frac{1}{2}k_{b_1}khL(p, v)^\top\mathcal{D}_{v_0}(R_0)^\top\Gamma(R_0)\epsilon_1, \qquad (21)$$

then define the hybrid system $\mathcal{H}_1 := (C_1, F_1, D_1, G_1)$ as follows:

$$\dot{x}_1 \in F_1(x_1) := \begin{pmatrix} F_d(r)\\ v_0\\ -R(q_1)R_0e_3\dfrac{T(\mu)}{m} + ge_3 - p_d^{(2)} + L(p, v)b\\ \dfrac{1}{2}\Pi(q_1)R_0(\omega - \omega_0)\\ k_zL(p, v)^\top\mathcal{D}_{v_0}\left(\overline{V}_0(p_0, v_0)\right)^\top\\ 0\\ \dfrac{1}{2}k_{b_1}khL(p, v)^\top\mathcal{D}_{v_0}(R_0)^\top\Gamma(R_0)\epsilon_1\end{pmatrix}$$

$$x_1 \in C_1 := \{x_1 \in \mathcal{X}_1 : (q_1, h) \in Q_\delta^+\}, \qquad (22a)$$

$$x_1^+ \in G_1(x_1) := (r, p_0, v_0, q_1, z, -h, \widetilde{b}_1)$$

$$x_1 \in D_1 := \{x \in \mathcal{X}_1 : (q_1, h) \in Q_\delta^-\}, \qquad (22b)$$

for some $\delta \in (0, 1)$, where $T(\mu) = m|\mu|$ and $\omega = \omega_1$, given by $\omega_1 := \omega_{0,1} + R_0{}^\top\left(-\omega_1^\star - k_qh\epsilon_1\right)$, with

$$\omega_1^\star := \frac{2k_zk_{V_0}}{kh}\left(\eta_1S(\mu) - S(\mu)S(\epsilon_1)\right)\mathcal{D}_{v_0}(V_0)^\top. \qquad (23)$$

$V_0$ given in (A.2) and $h \in H$ is a logic variable that enables controller switching and $H := \{-1, 1\}$ is a discrete set endowed with the discrete topology, but it can be regarded as a subset of $\mathbb{R}$ with the subspace topology. In particular, if we consider any function $V : \mathbb{R}^n \times H \to \mathbb{R}$ such that the map $x \mapsto V(x, h)$ is continuous for each $h \in H$, then $V(x, h)$ is continuous on $\mathbb{R} \times H$. This fact is used in the proof of the following theorem.

**Theorem 2.** *Let Assumption 1 hold. Then, there exists $K > |b|_\infty$ such that, for each $k_p, k_v, k_{V_0}, k_q, k, k_z, k_{b_1} > 0$, the solutions to the hybrid system (22) are bounded and the set $\mathcal{A}_1 := \{x_1 \in \mathcal{X}_1 : p_0 = 0, v_0 = 0, q_1 = (h, 0)\}$ is globally asymptotically stable.*

**Proof.** First of all, we prove that the hybrid system (22) meets the hybrid basic conditions (as given in Goebel et al., 2012): (1) since $\phi(x_1) := h\eta_1$ is continuous, the pre-image of closed sets under $\phi$ is also closed, thus both $C_1$ and $D_1$ are closed; (2) since $F_1(x_1)$ given in (22a), is a single valued function and continuous, it is locally bounded, convex and outer semicontinuous; (3) by Goebel et al. (2012, Lemma 5.10), the jump map $G_1(x_1)$ is outer semicontinuous if and only if $D_1 \times G_1(D_1)$ is closed. Notice that the jump map changes the logic variable but not the states, therefore $G_1(D_1)$ is closed and $G_1(x_1)$ is locally bounded for each $x_1 \in D_1$. Since $D_1$ is closed, we conclude that the jump map is outer-semicontinuous. Next, let us prove that every maximal solution to $\mathcal{H}_1$ is precompact, i.e. complete and bounded. Consider the following definition

$$V_1(x_1) := k_{V_0} V_0(p_0, v_0, z) + 2k(1 - h\eta_1) + \frac{1}{2k_{b_1}} \widetilde{b}_1{}^\top \widetilde{b}_1. \tag{24}$$

From the properties of $V_0$ and knowing that both $H$ and $\mathbb{S}^3$ are compact we have that for any $c > 0$, $V_1^{-1}(c)$ is compact. From Assumption 1 we have that the reference trajectory $r$ belongs to a compact set $\Omega_r$ and, since $q_0$ belongs to the compact set $\mathbb{S}^3$, then for any initial condition $(r, p_0, v_0, q_1, z, h, \widetilde{b}_1)(0, 0)$ we have that the set $U_1 := \{x_1 \in \mathcal{X}_1 : V_1(x_1) \le V_1(x_1(0, 0))\}$, is compact. We have that the time derivative of $V_1$ is given by

$$\left\langle \mathcal{D}_{x_1}(V_1), F_1(x_1)\right\rangle = -k_{V_0} k_z W_0(p_0, v_0) - k k_q \epsilon_1{}^\top \epsilon_1. \tag{25}$$

Defining

$$u_{c_1}(x_1) := \begin{cases} -k_{V_0} k_z W_0(p_0, v_0) - k k_q \epsilon_1{}^\top \epsilon_1 & \text{if } x_1 \in C_1 \\ -\infty & \text{otherwise,} \end{cases} \tag{26}$$

it is straightforward to see that $\left\langle \mathcal{D}_{x_1}(V_1), F_1(x_1)\right\rangle = u_{c_1}(x_1) \le 0$, for all $x_1 \in C_1 \cap U_1$. If $x_1 \in U_1 \cap D_1$ then $V_1(G_1(x_1)) - V_1(x_1) = 4kh\eta_1$. From (22b), we have that $h\eta_1 \le -\delta$ thus $V_1(G_1(x_1)) - V_1(x_1) \le -4k\delta$. Defining

$$u_{d_1}(x_1) = \begin{cases} -4k\delta & \text{if } x \in D_1 \\ -\infty & \text{otherwise,} \end{cases} \tag{27}$$

we have that $V_1(G_1(x_1)) - V_1(x_1) = u_{d_1}(x_1) < 0$ for all $x_1 \in U_1 \cap D_1$. These results show that any solution $x_1(t, j)$ to $\mathcal{H}_1$ remains in $U_1$ for all $(t, j) \in \operatorname{dom} x$. This together with the fact that $G_1(D_1) \subset C_1$ implies the completeness (from Goebel et al., 2012, Proposition 2.10) and the boundedness of solutions. Additionally, the relation $\overline{\operatorname{rge} x_1} \subset U_1$ is also verified and the growth of $V_1$ along solutions to $\mathcal{H}_1$ is bounded by $u_{c_1}, u_{d_1}$ on $U_1$. Then, since $\mathcal{H}_1$ satisfies the hybrid basic conditions and $V_1$ is continuous, by Goebel et al. (2012, Theorem 8.2), the precompact solutions $x_1$ to $\mathcal{H}_1$ approach the largest weakly invariant set $\mathcal{M}_1$ inside

$$V_1^{-1}(c) \cap U_1 \cap \left[\overline{u_{c_1}^{-1}(0)} \cup \left(u_{d_1}^{-1}(0) \cap G(u_{d_1}^{-1}(0))\right)\right], \tag{28}$$

for some $c > 0$. Since $u_{d_1}^{-1}(0) = \emptyset$ we have that, in particular, $\mathcal{M}_1 \subset \overline{u_{c_1}^{-1}(0)}$, with

$$u_{c_1}^{-1}(0) = \{x_1 \in \mathcal{X}_1 : p_0 = v_0 = 0, q_1 = (h, 0)\} = \mathcal{A}_1. \tag{29}$$

Since each maximal solution to $\mathcal{H}_1$ is precompact, it converges to $\mathcal{A}_1$. We conclude that $\mathcal{A}_1$ is globally attractive for the closed-loop hybrid system (22). Since $V_1$ is positive-definite relative to $\mathcal{A}_1$ and non-increasing along solutions to (22), then $\mathcal{A}_1$ is globally stable for the closed-loop hybrid system. Hence, we conclude that it is globally asymptotically stable. □

In the next section, we take advantage of Theorem 2 and backstepping techniques in order to solve Problem 1.

## 6. Global asymptotic stabilization of the full dynamic system

In this section, we develop a hybrid feedback law that is obtained from that of the previous section by means of backstepping. As before, due to the unknown disturbance $b$, the derivative of $\omega_1$ has to be estimated using a second estimator for $b$, denoted by $b_2 \in \mathbb{R}^3$. Let $\widetilde{\omega} := \omega - \omega_1$, $\widetilde{b}_2 := b_2 - b$ and

$$M = S(\omega) J\omega + J(\dot{\omega}_{1,2} + u), \tag{30}$$

where $k_\omega > 0$, $u \in \mathbb{R}^3$ denotes a new virtual input variable and $\dot{\omega}_{1,2} := \mathcal{D}_t(\omega_1)|_{b=b_2}$ denotes the estimate of $\dot{\omega}_1$ when using the estimate $b_2$. The difference between $\dot{\omega}_{1,2}$ and $\dot{\omega}_1$ is given by $\dot{\omega}_{1,2} - \dot{\omega}_1 = \mathcal{D}_{v_0}(\omega_1) L(p, v)\widetilde{b}_2$. Let $x_2 := (x_1, \widetilde{\omega}, \widetilde{b}_2)$ belong to $\mathcal{X}_2 := \mathcal{X} \times \mathbb{R}^3 \times \mathbb{R}^3$. Then, replacing (30) into (4b), we obtain $\dot{\omega} = u + \dot{\omega}_{1,2}$, allowing us to define the hybrid system $\mathcal{H}_2 := (C_2, F_2, D_2, G_2)$ as follows:

$$\dot{x}_2 \in F_2(x_2) := \begin{pmatrix} F_1(x_1) \\ -k_\omega \widetilde{\omega} + \mathcal{D}_{v_0}(\omega_1) L(p, v)\widetilde{b}_2 - khR_0{}^\top \epsilon_1 \\ -L(p, v)^\top \mathcal{D}_{v_0}(\omega_1)^\top \widetilde{\omega} \end{pmatrix}$$

$$x_2 \in C_2 := \{x_2 \in \mathcal{X}_2 : (q_1, h) \in Q_\delta^+\} \tag{31a}$$

$$x_2^+ \in G_2(x_2) := (G_1(x_1), \widetilde{\omega}, \widetilde{b}_2)$$

$$x \in D_2 := \{x_2 \in \mathcal{X}_2 : (q_1, h) \in Q_\delta^-\} \tag{31b}$$

where we have used

$$u := -k_\omega \widetilde{\omega} - khR_0{}^\top \epsilon_1, \tag{32a}$$

$$\dot{b}_2 := -L(p, v)^\top \mathcal{D}_{v_0}(\omega_1)^\top \widetilde{\omega}. \tag{32b}$$

With these definitions, we are able to state the main result of this paper.

**Theorem 3.** *Let Assumption 1 hold. Then, there exists $K > |b|_\infty$ such that, for each $k_p, k_v, k_{V_0}, k_q, k, k_z, k_{b_1}, k_{b_2} > 0$, the solutions to the hybrid system (31) are bounded and the set $\mathcal{A}_2 := \{x_2 \in \mathcal{X}_2 : x_1 \in \mathcal{A}_1, \widetilde{\omega} = 0\}$, is globally asymptotically stable.*

**Proof.** The proof of this theorem follows very closely the proof of Theorem 2. Namely, the proof that $\mathcal{H}_2$ meets the hybrid basic conditions is essentially the same so we dismiss it here.

Let us define the following Lyapunov function candidate

$$V_2(x_2) := k_{V_0} V_0(p_0, v_0, z) + 2k(1 - h\eta_1) + \frac{1}{2}\widetilde{\omega}^\top \widetilde{\omega} + \frac{1}{2k_{b_1}} \widetilde{b}_1{}^\top \widetilde{b}_1 + \frac{1}{2k_{b_2}} \widetilde{b}_2{}^\top \widetilde{b}_2. \tag{33}$$

Since $V_0$ is positive-definite relative to $\{(p_0, v_0, z) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^\ell : p_0 = 0, v_0 = 0, z = \Sigma_K^{-1}(b)\}$ and, by Assumption 1, $r \in \Omega_r$ for some compact set $\Omega_r$ we conclude that, for each initial condition $x_2(0, 0)$, the set $U_2 := \{x_2 \in \mathcal{X}_2 : V_2(x_2) \le V_2(x_2(0, 0))\}$, is compact. Using (25) and (31a) we have that the time derivative of $V_2$ is given by

$$\left\langle \mathcal{D}_{x_2}(V_2)^\top, F_2(x_2)\right\rangle = -k_{V_0} k_z W_0(p_0, v_0) - k k_q \epsilon_1{}^\top \epsilon_1 - k_\omega \widetilde{\omega}^\top \widetilde{\omega}. \tag{34}$$

Defining

$$u_{c_2}(x_2) := \begin{cases} -k_{V_0} k_z W_0 - k k_q \epsilon_1{}^\top \epsilon_1 - k_\omega \widetilde{\omega}^\top \widetilde{\omega} & \text{if } x_2 \in C_2 \\ -\infty & \text{otherwise} \end{cases} \tag{35}$$

it is straightforward to see that $\left\langle \mathcal{D}_{x_2}(V_2)^\top, F_2(x_2)\right\rangle = u_{c_2}(x_2) \le 0$, for all $x_2 \in C_2 \cap U_2$. From (31b), we have $h\eta_1 \le -\delta$, thus the following holds:

$$V_2(G_2(x_2)) - V_2(x_2) \le -4k\delta \quad \forall x_2 \in D_2. \tag{36}$$

Defining

$$u_{d_2}(x_2) = \begin{cases} -4k\delta & \text{if } x_2 \in D_2 \\ -\infty & \text{otherwise,} \end{cases} \tag{37}$$

we have that $V_2(G_2(x_2)) - V(x_2) = u_{d_2}(x_2) < 0$ for all $x_2 \in U_2 \cap D_2$. These results show that any given solution $x_2(t,j)$ to $\mathcal{H}_2$ remains in $U_2$ for all $(t,j) \in \operatorname{dom} x_2$. This together with the fact that $G_2(D_2) \subset C_2$ implies the completeness (from Goebel et al., 2012, Proposition 2.10) and the boundedness of solutions. Additionally, the relationship $\overline{\text{rge} x_2} \subset U_2$ is also verified and the growth of $V_2$ along solutions to $\mathcal{H}_2$ is bounded by $u_{c_2}, u_{d_2}$ on $U_2$. Then, since $\mathcal{H}_2$ satisfies the hybrid basic conditions and $V_2$ is continuous, by Goebel et al. (2012, Theorem 8.2) or Sanfelice, Goebel, and Teel (2007, Theorem 4.7), the precompact solutions to (31) approach the largest weakly invariant set $\mathcal{M}_2$ inside

$$V_2^{-1}(r) \cap U_2 \cap \left[\overline{u_{c_2}^{-1}(0)} \cup \left(u_{d_2}^{-1}(0) \cap G(u_{d_2}^{-1}(0))\right)\right], \tag{38}$$

for some $r > 0$. Since $u_{d_2}^{-1}(0) = \emptyset$ we have that, in particular, $\mathcal{M}_2 \subset \overline{u_{c_2}^{-1}(0)}$, with $u_{c_2}^{-1}(0) = \mathcal{A}_2$. Since each maximal solution to $\mathcal{H}_2$ is precompact, it converges to $\mathcal{A}_2$. We conclude that $\mathcal{A}_2$ is globally attractive for the hybrid system $\mathcal{H}_2$. Since $V_2$ is positive-definite relative to $\mathcal{A}_2$ and non-increasing along solutions to (31), then $\mathcal{A}_2$ is globally stable for the closed-loop hybrid system. Hence, we conclude that $\mathcal{A}_2$ is globally asymptotically stable for (31). It follows from the fact that solutions to the hybrid system remain in $U_2$ that $\widetilde{b}_2$ is bounded. □

To solve Problem 1, one needs to drop the assumption that $q_1$ is directly measured, and consider that $R(q_1)$ is measured instead. In this direction, let us define the controller variables $x_c := (z, h, b_1, b_2, \widehat{q}_1) \in \mathcal{X}_c$ for (9), where $\widehat{q}_1 \in \mathbb{S}^3$ is a memory variable that is part of the robust path-lifting strategy introduced in Mayhew et al. (2013) and $\mathcal{X}_c := \mathbb{R}^\ell \times H \times \mathbb{R}^\ell \times \mathbb{R}^\ell \times \mathbb{S}^3$. Then, we define the hybrid controller (9) as follows:

$$\begin{aligned}
F_c(x) &:= \begin{pmatrix} k_z L(p,v)^\top \mathcal{D}_{v_0}\left(\overline{V}_0(p_0,v_0)\right)^\top \\ 0 \\ \frac{1}{2} k_{b_1} k h L(p,v)^\top \mathcal{D}_{v_0}(R_0)^\top \Gamma(R_0)\epsilon_1 \\ -L(p,v)^\top \mathcal{D}_{v_0}(\omega_1)^\top \widetilde{\omega} \\ 0 \end{pmatrix} \\
\forall x \in C_c &:= \{x \in \mathcal{X} : (\Phi(\widehat{q}_1, RR_0^\top), h) \in Q_\delta^+, \\
&\qquad \operatorname{dist}(\widehat{q}_1, \mathcal{Q}(RR_0^\top)) \le \alpha\}, \\
G_c(x) &:= \begin{cases} (z, -h, b_1, b_2, \widehat{q}_1) & \text{if } x \in D_1 \setminus D_2, \\ \{(z, -h, b_1, b_2, \widehat{q}_1), & \\ (z, h, b_1, b_2, \Phi(\widehat{q}_1, RR_0^\top))\} & \text{if } x \in D_1 \cap D_2, \\ (z, h, b_1, b_2, \Phi(\widehat{q}_1, RR_0^\top)) & \text{if } x \in D_2 \setminus D_1 \end{cases} \\
\forall x \in D_c &:= D_1 \cup D_2,
\end{aligned} \tag{39}$$

where $0 < \alpha < 1$, $\mathcal{Q}(R)$ denotes the set of quaternions $\{q, -q\} \subset \mathbb{S}^3$ satisfying $R(q) = R(-q) = R$ for each $R \in SO(3)$,

$$\begin{aligned}
D_1 &:= \{x \in \mathcal{X} : (\Phi(\widehat{q}_1, RR_0^\top), h) \in Q_\delta^-, \\
&\qquad \operatorname{dist}(\widehat{q}_1, \mathcal{Q}(RR_0^\top)) \le \alpha\}, \\
D_2 &:= \{x \in \mathcal{X} : \operatorname{dist}(\widehat{q}_1, \mathcal{Q}(RR_0^\top)) \ge \alpha\},
\end{aligned} \tag{40}$$

$\operatorname{dist}(p, Q) := \inf\{p^\top q : q \in Q\}$ for each $p \in \mathbb{S}^3$ and $Q \subset \mathbb{S}^3$, and

$$\Phi(q, R) := \arg\max_{p \in \mathcal{Q}(R)} q^\top p,$$

is such that $q_1 = \Phi(\widehat{q}_1, RR_0^\top)$. Moreover, the output of the controller is

$$\begin{aligned}
T &= m\,|\mu|\,, \\
M &= S(\omega) J\omega + J^{-1}(-k_\omega \widetilde{\omega} - khR_0^\top \epsilon_1 + \dot{\omega}_{1,2}).
\end{aligned} \tag{41}$$

Backtracking the definitions of the variables in (41), it follows from Mayhew et al. (2013, Theorem 9) and Theorem 3 that (10) is globally asymptotically stable for the interconnection between (4) and (39). In the next section we present some experimental results that show the behavior of the closed-loop system using the given controller.

## 7. Experimental results

In order to experimentally evaluate the performance of the controller described in Section 6, we make use of the following components: (1) *Blade mQX quadrotor* (Horizon Hobby Inc., 2012), (2) VICON Bonita motion capture system (VICON, 2012), (3) MATLAB/Simulink software, and (4) custom made RF interface. The *Blade mQX quadrotor* weighs 80 g and has a radius of approximately 11 cm. This vehicle accepts the thrust and the angular velocity as inputs. It is readily available in the market and allows for easy integration with the other components of the control architecture. For more details on the system architecture and identification, the reader is referred to Cabecinhas, Cunha, and Silvestre (2014).

In order to assess that the hybrid controller was working as intended, we carried out the following experiment: (1) Set the desired position trajectory to (42); (2) Set the initial yaw of the quadrotor to be approximately 180 degrees away from the desired orientation; (3) Run the experiment for $h(0,0) = 1$ and $h(0,0) = -1$. Let us consider that $L(p,v) = I_3$ and $\sigma_K(s) = \frac{2K}{\pi}\arctan(s)$. Moreover, we have chosen the controller parameters $k_{V_0} = 0.01$, $k_z = 0.3$, $k_p = 3$, $k_v = 6$, $k = 3$, $K = 1$, $k_\omega = 40$, $k_{b_2} = 1$ and $k_{b_1} = 1$. The controller parameters for the position subsystem were obtained using LQR synthesis techniques and, even though, the performance attained by the LQR controller does not translate to this application due to saturation, it works fairly well when the tracking error is small. Also, one must select the attitude controller gains much higher that the position controller gains to ensure that the commanded orientation $R_0$ is closely tracked. The reference trajectory is given by

$$\begin{aligned}
p_d(t) &:= \begin{bmatrix} a\sin(2\pi\nu_0 t) & a\cos(2\pi\nu_0 t) & -h_0 \end{bmatrix}^\top, \\
R_d &:= I_3,
\end{aligned} \tag{42}$$

where $a = 1$ m, $f_0 = (2\pi)^{-1}$ Hz and $h_0 = 1$ m. This trajectory satisfies Assumption 1.

In this experiment, we test specifically the hybrid nature of the proposed controller since, if working as intended, different values of the logic variable produce different outcomes when the quadrotor is near a rotation error of 180 degrees. From the analysis of Fig. 1, it is possible to verify that this is indeed the case. For the experiment where $h(0,0) = -1$, the initial yaw angle is approximately 175° (or −185° if we subtract 360°) and it is quickly brought to zero. On the other hand, for the experiment where $h(0,0) = 1$, the initial yaw angle is 174° and, even though the initial yaw angles are only 1° apart, the quadrotor corrects its yaw angle by rotating in opposite directions. It may also be verified that this correction of the yaw angle has nothing but a small effect on the pitch and the roll angles. By introducing a hysteresis gap around rotations of 180°, where the behavior of the vehicle depends on the value of a logic variable, the hybrid controller reduces the possibility of chattering due to noise. This feature of the hybrid quaternion-based feedback is discussed in more detail in Mayhew et al. (2011a).

In Fig. 2, we present a comparison between the reference and the actual position of the vehicle for the two experiments. We see that there is little mismatch between the trajectories in both experiments and that the controller is able to converge to the given trajectory within the first 10 s of the experiments. The (component-wise) position tracking errors do not exceed 10 cm

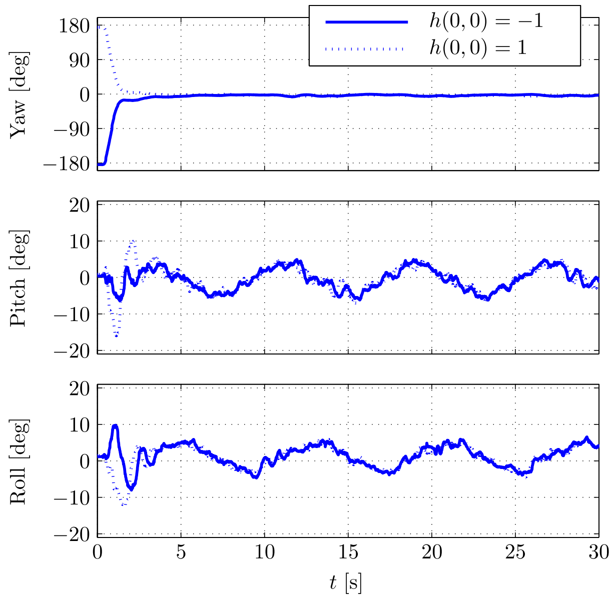


**Fig. 1.** Euler angles for two experiment runs starting roughly with the same initial.

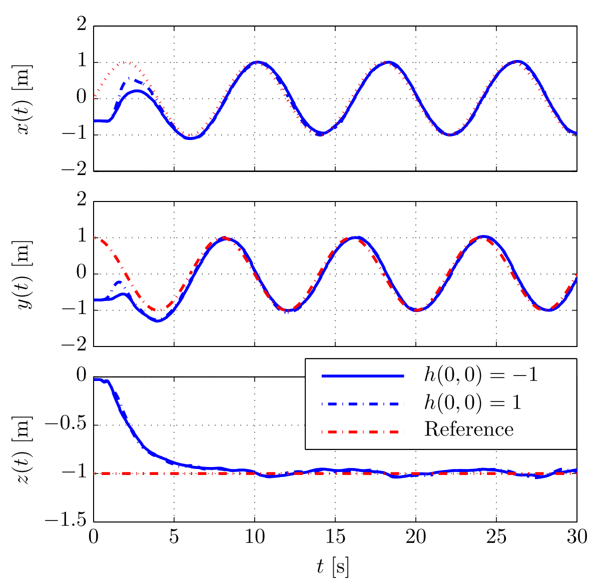


**Fig. 2.** Comparison between the position of the quadcopter and the reference trajectory for two experimental runs with initial yaw error of approximately 180° and different initial values of $h(0, 0)$.

after the first 10 s, which we attribute mostly to delays in the system, since it is possible to see that there is some lag in the tracking of the given trajectory.

From Figs. 1 and 2 it can be concluded that the controller proposed in this paper yields very good results, despite the very simple model that was considered in its design.

## 8. Conclusion

In this paper, we designed a quaternion-based hybrid controller that globally asymptotically stabilizes a class of underactuated vehicles to a smooth reference position trajectory. In particular, the proposed controller is robust to bounded state dependent acceleration disturbances and small measurement noise. Moreover, the proposed controller also minimizes the angle to a reference orientation. The proposed controller was tested in an experimental setup using a optical motion capture system.


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## Appendix. Proof of Lemma 1

In what follows, let $\overline{u}_0(p_0, v_0) := -\Sigma_K(k_p p_0 + k_v v_0)$. For each $K, k_p, k_v > 0$ there exists a positive definite and symmetric matrix $P \in \mathbb{R}^{2\times 2}$ such that

$$\left\langle \mathcal{D}_{(p_0,v_0)}\left(\overline{V}_0(p_0, v_0)\right)^\top, [v_0{}^\top\ \overline{u}_0(p_0, v_0)^\top]^\top \right\rangle < 0,$$

for each $(p_0, v_0) \in \mathbb{R}^6 \setminus \{0\}$ and

$$\left\langle \mathcal{D}_{(p_0,v_0)}\left(\overline{V}_0(p_0, v_0)\right)^\top, [v_0{}^\top\ \overline{u}_0(p_0, v_0)^\top]^\top \right\rangle = 0,$$

for $(p_0, v_0) = 0$ (cf. Casau et al., 2013, Proposition 1).

It follows from (13) and from the reverse triangle inequality that

$$|\mu(r, p_0, v_0, z)| \geq g - \left|\Sigma_K(k_p p_0 + k_v v_0)\right| - |L(p, v)\,\Sigma_K(z)| - \left|p_d^{(2)}\right|.$$

Since $|\Sigma_K(x)| \leq \sqrt{n}K$ for each $x \in \mathbb{R}^n$ and $|Ax| \leq |A|_2\,|x|$ for each $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$, it follows that

$$|\mu(r, p_0, v_0, z)| \geq g - (\sqrt{3} + \sqrt{\ell} \sup_{(p,v)\in\mathbb{R}^6} |L(p, v)|_2)K - \sup_{r\in\Omega_r}\left|p_d^{(2)}\right|.$$

Then, it follows from (7) that it is possible to select $K > |b|_\infty$ such that $|\mu(r, p_0, v_0, z)| > 0$ for each $(p_0, v_0, z) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^\ell$. It follows from the triangle inequality that, given a reference trajectory satisfying Assumption 1 and disturbances $(p, v, b) \mapsto L(p, v)b$ for (11), if (7) holds for each $(p, v) \in \mathbb{R}^3$, then there exists $K > |b|_\infty$ such that

$$0 < |\mu(r, p_0, v_0, z)| \leq g + (\sqrt{3} + \sqrt{\ell} \sup_{(p,v)\in\mathbb{R}^6} |L(p, v)|_2)K + \sup_{r\in\Omega_r}\left|p_d^{(2)}\right|, \tag{A.1}$$

for each $(p_0, v_0, z) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^\ell$. It follows directly from the lower and upper bounds on $\mu(r, p_0, v_0, z)$ given previously that there exists $\overline{T} > 0$ such that $0 < T(\mu(r, p_0(t), v_0(t), z(t))) \leq \overline{T}$ for each $r \in \Omega_r$ and for each solution $t \mapsto (p_0(t), v_0(t), z(t))$, defined for each $t \geq 0$. Consider the following continuous function

$$V_0(p_0, v_0, z) = k_z\overline{V}_0(p_0, v_0) - b^\top z + \sum_{i=1}^{\ell}\left(\int_0^{|z_i|} \sigma_K(\xi)\, d\xi + \int_0^{|b_i|} \sigma_K^{-1}(\xi)\, d\xi\right). \tag{A.2}$$

From an application of Young's inequality it follows that

$$b_i z_i \leq |b_i|\,|z_i| \leq \int_0^{|z_i|} \sigma_K(\xi)\, d\xi + \int_0^{|b_i|} \sigma_K^{-1}(\xi)\, d\xi, \tag{A.3}$$

it is possible to conclude that (A.2) is positive definite relative to the compact set $\{(p_0, v_0, z) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^\ell : p_0 = v_0 = 0, z = \Sigma_K^{-1}(b)\}$. That $V_0$ is radially unbounded follows from the fact that

$$\lim_{|z_i|\to+\infty} \int_0^{|z_i|} \sigma_K(\xi)\, d\xi - b|z_i| = +\infty, \tag{A.4}$$

where $b > 0$ and $i \in \{1, 2, 3\}$. Suppose that the limit in (A.4) is finite, then

$$\lim_{|z_i|\to+\infty} \int_0^{|z_i|} \sigma_K(\xi)\, d\xi - b|z_i| = \lim_{|z_i|\to+\infty} \int_0^{|z_i|} \sigma_K(\xi)\, d\xi - b|z_i| + \lim_{|z_i|\to+\infty} |z_i|(\sigma_K(|z_i|) - b). \tag{A.5}$$

However, using the properties of the $K$-saturation function, since $K > |b|_\infty$ we have that $\lim_{|z_i|\to+\infty} |z_i|(\sigma_K(|z_i|) - b)$ does not converge, thus the limit in (A.4) cannot converge. By (A.3) we have that

$$\lim_{|z_i|\to+\infty} \int_0^{|z_i|} \sigma_K(\xi)\, d\xi - b|z_i| = +\infty. \quad \square$$

Since $\overline{V}_0$ is also radially unbounded, then $V_0$ is radially unbounded.

Let $F$ denote (15), then the time derivative of (A.2) is given by

$$\left\langle \mathcal{D}_{(p_0,v_0)} (V_0(p_0, v_0))^\top, F \right\rangle = -k_z W_0(p_0, v_0), \tag{A.6}$$

thus (A.6) is negative definite relative to $p_0 = v_0 = 0$. Since (A.2) is radially unbounded, for any initial condition $(p_0, v_0, z)(0)$ then the sub-level set

$$\begin{aligned}\Omega_0 := \{&(p_0, v_0, z) \in \Omega_r \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^\ell \\ &: V_0(p_0, v_0, z) \le V_0((p_0, v_0, z)(0))\}\end{aligned} \tag{A.7}$$

is compact. It follows from Khalil (2002, Theorem 4.8) that the set $\{(p_0, v_0, z) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^\ell : p_0 = v_0 = 0, z = \Sigma_K^{-1}(b)\}$ is globally stable for the system (11), and it follows from Khalil (2002, Theorem 8.4) that $V_0(p_0, v_0, z)$ converges to 0, therefore each solution converges to

$$\Omega_0 \cap \{(p_0, v_0, z) \in \Omega_r \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^\ell : p_0 = v_0 = 0\}, \tag{A.8}$$

which is a subset of $\mathcal{A}_0$.

## References


Aguiar, A. P., & Hespanha, J. P. (2007). Trajectory-tracking and path-following of underactuated autonomous vehicles with parametric modeling uncertainty. *IEEE Transactions on Automatic Control*, *52*(8), 1362–1379.

Betty, M. F. (1986). *Principles of engineering mechanics, Vol. 1*. Plenum Press.

Bhat, S. P., & Bernstein, D. S. (2000). A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon. *Systems & Control Letters*, *39*(1), 63–70.

Bullo, F., & Lewis, A. D. (2005). *Texts in applied mathematics, Geometric control of mechanical systems*. Springer.

Cabecinhas, D., Cunha, R., & Silvestre, C. (2014). A nonlinear quadrotor trajectory tracking controller with disturbance rejection. *Control Engineering Practice*, *26*, 1–10.

Casau, P., Sanfelice, R.G., Cunha, R., Cabecinhas, D., & Silvestre, C. (2013). Global trajectory tracking for a class of underactuated vehicles. In *American control conference, ACC* (pp. 419–424).

Casau, P., Sanfelice, R.G., Cunha, R., Cabecinhas, D., & Silvestre, C. (2014). Optimal trajectory tracking for a class of underactuated vehicles. In *International conference on robotics and automation*.

Chaturvedi, N., Sanyal, A., & McClamroch, N. (2011). Rigid-body attitude control. *IEEE Control Systems Magazine*, *31*(3), 30–51.

Fjellstad, O.-E., & Fossen, T. (1994). Quaternion feedback regulation of underwater vehicles. In *Proceedings of the third IEEE conference on control applications, Vol. 2* (pp. 857–862).

Frazzoli, E., Dahleh, M., & Feron, E. (2000). Trajectory tracking control design for autonomous helicopters using a backstepping algorithm. In *Proceedings of the American control conference, Vol. 6* (pp. 4102–4107).

Goebel, R., Sanfelice, R. G., & Teel, A. R. (2012). *Hybrid dynamical systems: modeling, stability and robustness*. Princeton University Press.

Goodarzi, F., Lee, D., & Lee, T. (2013). Geometric nonlinear PID control of a quadrotor UAV on SE(3). ArXiv e-prints.

Herrick, K. (2000). Development of the unmanned aerial vehicle market: forecasts and trends. *Air & Space Europe*, *2*(2), 25–27.

Horizon Hobby Inc., 2012. Blade mQX ultra micro quad-copter. URL: http://www.bladehelis.com/.

Hua, M.-D., Hamel, T., Morin, P., & Samson, C. (2009). A control approach for thrust-propelled underactuated vehicles and its application to VTOL drones. *IEEE Transactions on Automatic Control*, *54*(8), 1837–1853.

Joshi, S., Kelkar, A., & Wen, J.-Y. (1995). Robust attitude stabilization of spacecraft using nonlinear quaternion feedback. *IEEE Transactions on Automatic Control*, *40*(10), 1800–1803.

Khalil, H. K. (2002). *Nonlinear systems*. Prentice Hall.

Kinsey, J.C., Eustice, R.M., & Whitcomb, L.L. (2006). A survey of underwater vehicle navigation: Recent advances and new challenges. In *IFAC conference of manoeuvering and control of marine craft, Lisbon, Portugal. Invited paper*.

Koditschek, D. E. (1989). The application of total energy as a Lyapunov function for mechanical control systems. In *Contemporary mathematics*: *Vol. 97*. American Mathematical Society.

Kristiansen, R., Nicklasson, P., & Gravdahl, J. (2009). Satellite attitude control by quaternion-based backstepping. *IEEE Transactions on Control Systems Technology*, *17*(1), 227–232.

Lee, T., Leok, M., & Harris McClamroch, N. (2011). Nonlinear robust tracking control of a quadrotor UAV on SE(3). ArXiv e-prints.

Levine, J., & Müllhaupt, P. (2011). *Advances in the theory of control, signals and systems with physical modeling*. Springer.

Li, S., Ding, S., & Li, Q. (2010). Global set stabilization of the spacecraft attitude control problem based on quaternion. *International Journal of Robust and Nonlinear Control*, *20*(1), 84–105.

Magnus, J. R., & Neudecker, H. (1985). Matrix differential calculus with applications to simple, Hadamard, and Kronecker products. *Journal of Mathematical Psychology*, *29*(4), 474–492.

Mayhew, C., Sanfelice, R., & Teel, A. (2011a). Quaternion-based hybrid control for robust global attitude tracking. *IEEE Transactions on Automatic Control*, *56*(11), 2555–2566.

Mayhew, C., Sanfelice, R., & Teel, A. (2011b). Synergistic Lyapunov functions and backstepping hybrid feedbacks. In *American control conference, ACC* (pp. 3203–3208).

Mayhew, C., Sanfelice, R., & Teel, A. (2013). On path-lifting mechanisms and unwinding in quaternion-based attitude control. *IEEE Transactions on Automatic Control*, *58*(5), 1179–1191.

Mayhew, C. G., & Teel, A. R. (2011a). On the topological structure of attraction basins for differential inclusions. *Systems & Control Letters*, *60*(12), 1045–1050.

Mayhew, C.G., & Teel, A.R. (2011b). Synergistic potential functions for hybrid control of rigid-body attitude. In *American control conference, ACC* (pp. 875–880).

Padfield, G. (2007). *Helicopter flight dynamics: the theory and application of flying qualities and simulation modelling*. Blackwell Publishing.

Sanfelice, R., Goebel, R., & Teel, A. (2007). Invariance principles for hybrid systems with connections to detectability and asymptotic stability. *IEEE Transactions on Automatic Control*, *52*(12), 2282–2297.

Shuster, M. D. (1993). A survey of attitude representations. *The Journal of the Astronautical Sciences*, *41*(4), 439–517.

Tayebi, A., & McGilvray, S. (2006). Attitude stabilization of a VTOL quadrotor aircraft. *IEEE Transactions on Control Systems Technology*, *14*(3), 562–571.

VICON, 2012. Motion capture systems from vicon. URL: http://www.vicon.com.

Wertz, J. R. (1978). *Spacecraft attitude determination and control*. Kluwer Academic Publishers.

Wisniewski, R., & Kulczycki, P. (2003). Rotational motion control of a spacecraft. *IEEE Transactions on Automatic Control*, *48*(4), 643–646.

Zhao, S., Dong, W., & Farrell, J.A. (2013). Quaternion-based trajectory tracking control of VTOL-UAVs using command filtered backstepping. In *American control conference, ACC* (pp. 1018–1023).



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