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Position tracking for underactuated rigid bodies on $SE(3)$

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Abstract

The problem of tracking a desired trajectory is of fundamental importance in real applications where some (robotic) system is required to follow a pre-planned or pre-specified path with time constraints. For underactuated systems this problem is not always solvable since the desired trajectory may not belong to the set of feasible trajectories for the given system. However real life applications often only require tracking of some of the variables, the most common example being a unicycle type robot following a preassigned 2D path. In this paper we study the problem of position tracking for an underactuated rigid body in $SE(3)$.

Keywords: Trajectory tracking, Nonlinear Control, Nonholonomic Systems, Underactuated Systems, Differential Geometry

1. Introduction

Tracking a desired trajectory is a frequent problem is control and robotics, where a pre-planned path representing the accomplishment of certain goals must be enforced. This pre-specified path may represent an optimal solution for the problem, a required maneuver to be executed such as docking of a vehicle, or the outcome of some higher-level controller.

For fully actuated systems this problem is now well understood and solutions are proposed in standard textbooks on nonlinear control [12], [16] and [18]. On the other hand tracking for underactuated systems is a challenging problem from the theoretical point of view since not all trajectories are feasible by the system, and the results developed for fully actuated systems fail to apply. From the practical point of view this problem is also of great importance since the development of systems with less actuators allows for reductions in the cost of the overall system and in full actuated systems represents a valuable safeguard regarding malfunctioning of some of the available actuators.

In this article we will address a special class of this problem where the system in only required to track some of the state variables, more specifically we will consider underactuated rigid bodies on the special euclidean group $SE(3)$ where it is only required that the system tracks a reference position in three dimensional space. The importance of this problem comes from the fact that often a mission is only specified in terms of a desired position trajectory and no orientation information is available. This is the case for underwater vehicles such as benthic labs that must follow a predetermined path towards the areas to explore or unmanned aircrafts that must execute pre-specified surveillance missions. In fact the original motivation for this problem comes from unicycle[1] type vehicles moving in $SE(2)$ that must follow certain paths to accomplish team work tasks such as playing soccer (www.robocup.org). Our motivation has also compelled us to study only the kinematic version of the problem since it is sufficient for controlling unicycle type robots, extentions to dynamics where more difficult phenomena occur such as non-zero side slip will be delayed to future contributions.

Traditional approaches to this problem involve linearization about the reference trajectory and methods from linear control theory resulting in a global gain-scheduled control law [13] or linear time-varying control [21]. Search based methods to derive state variables and control input values necessary to track a desired reference are proposed in [4], [19], although only fully actuated systems are considered. Other approaches include adaptive and feedback linearization schemes [10], using constant forward speed, thereby reducing the problem to control the attitude of the rigid body towards the reference trajectory [9]. This approach was originally introduced in [17, 15] and since then more advanced techniques have also been applied to planar robots such as partial feedback linearization and state diffeomorphism (change of coordinates) and dynamic feedback linearization, [1] [20]. A survey of the various methods of control and trajectory tracking for mobile robots is given in [22] and for ocean vehicles in [10].

Contrary to the described approaches in this paper we will address the problem from a coordinate-free perspective, therefore allowing a simpler and more general understanding and presentation of the results often obscured by the particular choice of coordinates chosen. This is specially evident on the literature regarding the $SE(3)$ case, where the parameterizations may contain singularities (Euler angles) or more variables then the dimension of the parameterized space (Euler parameters or unit norm quaternions). This approach makes use of several techniques from differential geometry and has been strongly influenced by work on tracking with similar approaches such as [7, 5, 6]. A good introduction to nonholonomic systems in the context of Riemannian manifolds is given in [2].

The paper is divided as follows: in Section 2 mathematical results, concepts and notation used throughout the paper are introduced. In Section 3 the position tracking problem is mathematically formalized and smoothness and boundedness assumptions are presented. An intuitive explanation is given for the control strategy at subsection 3.3 which is rigorously studied in subsections 3.4 and 3.5. This subsection contain the main results of the paper. Section 4 contains the particularization of the already developed results for the $SE(2)$ case and simulation results follow in Section 5. Finally conclusions are addressed in Section 6 and an appendix contains some algebraic manipulations of the results presented in section 3.

2. Mathematical preliminaries

2.1. **The Lie group $SE(3)$.** Consider a three-dimensional rigid body freely moving in $\mathbb{R}^3$, an inertial frame I fixed in space and a body frame B fixed to the body. The natural state space for this system will be the set of all linear transformations from the frame I to the frame B, representing at each instant of time the configuration

[1] Also known as differential drive type robots on some literature.

(position and orientation) of the rigid body's frame B with respect to the inertial frame. This set is not only a differentiable manifold but also a group, therefore a Lie group [3], namely $SE(3)$:

$$SE(3) = \left\{ \begin{bmatrix} R & x \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \; : \; R \in \mathbb{R}^{3\times 3},\, x \in \mathbb{R}^3,\, R^{-1} = R^T,\, det(R) = 1 \right\} \tag{1}$$

The tangent space at the identity element of the group constitutes a Lie algebra, $\mathfrak{se}(3)$ defined as:

$$\mathfrak{se}(3) = \left\{ \begin{bmatrix} \Omega & v \\ 0 & 0 \end{bmatrix} \; : \; \Omega \in \mathbb{R}^{3\times 3},\, v \in \mathbb{R}^3,\, \Omega^T = -\Omega \right\} \tag{2}$$

Since an algebra implies a linear space structure, we shall consider in this paper the following base for $\mathfrak{se}(3)$:

$$X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

$$X_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

2.2. **Left invariant kinematics.** The left translation map L_g (on any Lie group) is defined as:

$$\begin{aligned} L_g &: G \to G \\ L_g(x) &= g \cdot x \end{aligned} \tag{5}$$

Using the above definition we can define left invariance of a vector field by requiring that a left invariant vector field X satisfies:

$$X(g) = X(g \cdot e) = TL_g X(e) \tag{6}$$

where e is the group identity and $TL_g : TG \to TG$ is the derivative of the map L_g, using this fact the underactuated rigid body kinematic equations take a special simple form:

$$\frac{d}{dt} g = g \cdot (X_1 u_1 + X_2 u_2 + X_3 u_3 + X_4 u_4) \tag{7}$$

where u_1, u_2, u_3 controls roll, pitch and yaw of the rigid body, respectively, and u_4 controls the forward velocity. Note that the system is underactuated since motion along the basis vectors X_5 and X_6 is not possible.

2.3. **Riemannian metrics.** A bilinear form on a vector space V over the real field is a multilinear map $\Theta : V \times V \to \mathbb{R}$, it is called symmetric if $\Theta(v,w) = \Theta(w,v)$ and positive definite if $\Theta(v,v) \geq 0$ and if equality holds iff $v = 0$. Θ in also called an inner product on the vector space V and denoted by $< .,. >$. If we now define an inner product at each point p of a manifold M, that is $\Theta_p : T_pM \times T_pM \to \mathbb{R}$ ensuring that Θ_p varies smoothly from point to point (on any coordinate chart the functions $g_{ij} = \Theta(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ are C^∞), we call Θ a Riemannian metric and M a Riemannian manifold.

In the special case where the manifold is a Lie group G we can define a metric at the identity element Θ_e and left translate it to the remaining points of the manifold [8] by requiring that:

$$< L_1(g), L_2(g) >_g = < gL_1, gL_2 >_g = < g^{-1}gL_1, g^{-1}gL_2 >_{g^{-1}g} = < L_1, L_2 >_e \tag{8}$$

for any left invariant vector fields L_1 and L_2.

In this paper we shall associate a vector $\{\omega, v\}$ with each vector field L on $SE(3)$ through the identification:

$$\{(\omega_1,\omega_2,\omega_3),(v_1,v_2,v_3)\} \longleftrightarrow L = \begin{bmatrix} 0 & -\omega_3 & \omega_2 & v_1 \\ \omega_3 & 0 & -\omega_1 & v_2 \\ -\omega_2 & \omega_1 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

This identification allows us to express a left invariant metric on $SE(3)$ by:

$$\Theta = \begin{bmatrix} \alpha\mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \beta\mathbf{I}_{3\times3} \end{bmatrix} \tag{10}$$

for a discussion on the possible metrics on $SE(3)$ and its relation with the kinematic connection we defer the reader to [23] and the references therein.

2.4. **Connections.** A C^∞ connection or covariant derivative ∇ on a fiber bundle E of rank n over a manifold M is a linear map $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ that verifies the Leibniz rule:

$$\nabla(fX) = \mathrm{d}f \otimes X + f\nabla X \quad \forall\, f \in C^\infty(M),\ X \in \Gamma(E) \tag{11}$$

Let $\varepsilon_1, ..., \varepsilon_n$ be a local basis for the fiber over M, we can completely determined the connection by defining an $n \times n$ matrix of covectors ω_j^k on M by:

$$\nabla\varepsilon_j = \omega_j^k \varepsilon_k \tag{12}$$

since for any section of E, $X = X^k\varepsilon_k$, $X^k \in C^\infty(M)$ its covariant derivative can be computed as:

$$\begin{aligned} \nabla X &= \nabla(X^k\varepsilon_k) \\ &= \mathrm{d}X^k \otimes \varepsilon_k + X^k\nabla\varepsilon_k \quad \text{by Leibniz rule} \\ &= (\mathrm{d}X^k + X^j\omega_j^k) \otimes \varepsilon_j \end{aligned} \tag{13}$$

In this paper we will only work with connections on the tangent bundle, so the definitions specialize to $\nabla : TM \to T^*M \otimes TM$, and on a coordinate chart this map is completely defined by the Christofell symbols:

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k} \tag{14}$$

Comparing (14) and (12) we realize that $\omega_j^k = \Gamma_{ij}^k dx_i$, therefore the connection can be locally characterized by:

$$\nabla_X Y = \Big(\frac{\partial Y^i}{\partial x^j} X^j + \Gamma_{ij}^k X^i Y^j\Big) \frac{\partial}{\partial x^k} \tag{15}$$

We say that a connection is torsion-free or symmetric if

$$[X, Y] = \nabla_X Y - \nabla_Y X \tag{16}$$

where $[X, Y]$ is the Lie bracket between the vector fields X and Y. Given a Riemannian metric $< .,. >$ we say that a connection is compatible with the metric if:

$$\mathcal{L}_X < Y, Z > = < \nabla_X Y, Z > + < X, \nabla_X Z > \tag{17}$$

Unfortunately a connection is not uniquely determined on a manifold however given a Riemannian metric, there is one and only one connection compatible with the metric and torsion-free. We defer the reader to [3] to further material regarding covariant differentiation and Riemannian geometry.

We shall be using the kinematics connection compatible with the previously given left-invariant metric and whose non zero Christofell symbols we reproduce here for completeness:

$$\begin{array}{ll} \Gamma^3_{12} = \Gamma^2_{31} = \Gamma^1_{23} = \frac{1}{2}, & \Gamma^3_{21} = \Gamma^2_{13} = \Gamma^1_{32} = -\frac{1}{2} \\ \Gamma^6_{15} = \Gamma^4_{26} = \Gamma^5_{34} = 1, & \Gamma^6_{24} = \Gamma^4_{35} = \Gamma^5_{16} = -1 \end{array} \tag{18}$$

2.5. **Error functions.** We shall define error functions on $\mathbb{R}^n$ since we are only interested in tracking trajectories in $\mathbb{R}^3$, for a definition of error functions on abstract manifolds the reader is deferred to [7]. An error function is a map $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, such that $\phi(x,r) \geq 0$ and $\phi(x,r) = 0$ iff $x = r$. We shall also impose that $\mathrm{d}_2\phi(x,r) = -\mathrm{d}_1\phi(x,r)$ where d_1 is the exterior derivative with respect to x and d_2 the exterior derivative with respect to r. This will allow the time derivative of $\phi(x,r)$ being expressed by the familiar expression:

$$\frac{d}{dt}\phi(x,r) = \mathrm{d}_1\phi(x,r)\cdot\dot{x} + \mathrm{d}_2\phi(x,r)\cdot\dot{r} = \mathrm{d}_1\phi(x,r)(\dot{x}-\dot{r}) \tag{19}$$

We shall say that the error function is (uniformly) quadratic lower bounded if there is a scalar $b \geq 0$ such that:

$$\phi(x,r) \geq b\|\mathrm{d}_1\phi(x,r)\|^2 \tag{20}$$

Note than in abstract manifolds this condition may only hold locally according to the topology of the manifold.

3. Tracking for nonholonomic systems

3.1. **Problem formulation.** The goal of this paper is to describe an algorithm to track a desired position reference $r(t)$ disregarding however the rigid body orientation. Mathematically the position tracking or position tracking is defined as:

A control law $u(g)$ solves the position tracking problem if:

- The tracking error $\phi(t) = \phi(x(t), r(t))$ and the controls u_i are bounded for all time.
- The tracking error asymptotically decays to zero, $\lim_{t\to\infty}\phi(t) = 0$

It is usual to include another requirement when only feasible trajectories are being tracked, namely that $\phi(0) = 0 \Rightarrow \phi(t) = 0$. However this requirement may not be satisfied if one wishes to track trajectories not feasible by all the states of the system. Suppose that $\phi(0) = 0$ and that $\frac{d}{dt}r(t) \notin Span\{X_1, X_2, X_3, X_4\}$ under this scenario one can never guarantee that the error function will remain zero.

3.2. **Regularity and boundedness assumptions.** We shall assume the following regularity and boundedness properties:

- $\phi(x,r) \in C^2$
- $r(t)$ is twice differentiable
- $\sup_{t\in\mathbb{R}} \|r(t)\| < \infty$
- $\sup_{t\in\mathbb{R}} \|\dot{r}(t)\| < \infty$
- $\sup_{t\in\mathbb{R}} \|\ddot{r}(t)\| < \infty$

We assume that the reference trajectory be twice differentiable, this is not a restrictive assumption since it is desirable that reference be as smooth as possible. Boundedness assumptions on the reference trajectory are also standard assumptions.

3.3. **Intuitive motivation.** To achieve exponential tracking of the rigid body position it would desirable that the vector field $\dot{x} = X$ could be chosen to be $X = \dot{r} - \lambda(\mathrm{d}_1\phi(x,r))^T$, where we use the metric $(g_{ij} = \beta\mathbf{I}_{3\times3})$ on $\mathbb{R}^3$ to transform the covector $\mathrm{d}_1\phi(x,r)$ in the vector $g^{ij}(\mathrm{d}_1\phi(x,r))_j = \frac{1}{\beta}(\mathrm{d}_1\phi(x,r))^i = \frac{1}{\beta}(\mathrm{d}_1\phi(x,r))^T = \lambda(\mathrm{d}_1\phi(x,r))^T$. Therefore, by using $V_1 = \phi(x,r)$ as a candidate Lyapunov function one imediatly sees that:

$$\begin{aligned} \frac{d}{dt}V_1 &= \mathrm{d}_1\phi(x,r)\cdot(\dot{x}-\dot{r}) \\ &= \mathrm{d}_1\phi(x,r)\cdot\left(\dot{r}-\lambda(\mathrm{d}_1\phi(x,r))^T-\dot{r}\right) \\ &= -\lambda\mathrm{d}_1\phi(x,r)\cdot(\mathrm{d}_1\phi(x,r))^T \end{aligned} \tag{21}$$

which is negative semi-definite, negative definiteness is a consequence of the quadratic nature of ϕ, in fact using the inequality in (20):

$$\frac{d}{dt}V_1 = -\lambda\mathrm{d}_1\phi(x,r)\cdot(\mathrm{d}_1\phi(x,r))^T = -\lambda\|\mathrm{d}_1\phi(x,r)\|^2 \le -\frac{\lambda}{b}\phi(x,r) \tag{22}$$

It is not always possible to freely assign the vector field $\dot{x}$ due to the kinematic restrictions of the system, however the above observation suggests the following approach to solve the problem:

- Use roll, pitch and yaw inputs to align the vector field $X_r = \begin{bmatrix} \mathbf{0}_{3\times3} & X \\ \mathbf{0}_{1\times3} & 0 \end{bmatrix}$ with X_4.
- Project X_r on X_4, to determine the forward velocity control input.

This approach will now be described in more detail.

3.4. **Orientation control.** To ensure that X_r belongs to $Span\{X_1,...,X_4\}$ one must derive a control law that stabilizes the system in the following set $\Psi = \{g \in SE(3)\ :< X_r, X_5(g) >_g = 0,\ < X_r, X_6(g) >_g = 0\}$.

We can build a candidate Lyapunov function measuring the "distance" to the set Ψ, let $\psi_i : \mathbb{R}\times\mathbb{R} \to \mathbb{R} \quad i = 1,2$ be two error functions and consider the following Lyapunov candidate function:

$$V_2 = \psi_1(< X_r, X_5 >, 0) + \psi_2(< X_r, X_6 >, 0) \tag{23}$$

After some tedious algebra presented in the appendix it can be shown that its time derivative is given by:

$$\begin{aligned} \frac{d}{dt}V_2 &= \mathrm{d}\psi_1\big(< \{0, \ddot{r} - \lambda\frac{d}{dt}(\mathrm{d}_1\phi(x,r))|^T_{x\,fixed}\}, X_5 > -\lambda u_4 < \frac{\partial^2\phi(x,r)}{\partial x^i x^j}X_4, X_5 > \\ &\quad + u_1 < X_r, X_6 > - u_3 < X_r, X_4 >\big) \\ &+ \mathrm{d}\psi_2\big(< \{0, \ddot{r} - \lambda\frac{d}{dt}(\mathrm{d}_1\phi(x,r))|^T_{x\,fixed}\}, X_6 > -\lambda u_4 < \frac{\partial^2\phi(x,r)}{\partial x^i x^j}X_4, X_6 > \\ &\quad + u_2 < X_r, X_4 > - u_1 < X_r, X_5 >\big) \end{aligned} \tag{24}$$

This means that if we choose $\phi(x,r)$ in such a way that $< \frac{\partial^2\phi(x,r)}{\partial x^i x^j}X_4, X_5 >= 0$ and $< \frac{\partial^2\phi(x,r)}{\partial x^i x^j}X_4, X_6 >= 0$ and as long as $< X_r, X_4 >\neq 0$ we can use u_2 and u_3 to exponentially steer the rigid body towards the set Ψ. Before stating this result we will give a more useful characterization of the allowed error functions:

Proposition 3.1. *The requirement that* $< \frac{\partial^2\phi(x,r)}{\partial x^i x^j}X_4, X_5 >= 0 =< \frac{\partial^2\phi(x,r)}{\partial x^i x^j}X_4, X_6 >$ *is satisfied iff* $\phi(x,r) = \frac{1}{2}k(r)(x-r)^T(x-r)$, *where* $k(r)$ *is a smooth function of* r.

Proof. $\frac{\partial^2\phi(x,r)}{\partial x^i x^j}X_4$ must not be a linear combination of X_5 and X_6 which in turn implies the following matrix equation $k(x,r)X_4 = \frac{\partial^2\phi(x,r)}{\partial x^i x^j}X_4$, where $k(x,r)$ is a smoothly varying scalar gain (a function on $\mathbb{R}^3\times\mathbb{R}^3$). The solution is clearly $\frac{\partial^2\phi(x,r)}{\partial x^i x^j} = k(x,r)\mathbf{I}_{3\times3}$. Integrating $\frac{\partial^2\phi(x,r)}{\partial x^1 x^2} = 0$ we get $\frac{\partial\phi(x,r)}{\partial x^1} = f_1(x^1,r) + f_2(x^3,r)$, note that f_1 or f_2 cannot be a function of both x^1 and x^3 because it would violate $\frac{\partial^2\phi(x,r)}{\partial x^1 x^3} = 0$. Integrating once more we get that $\phi(x,r) = h_1(x^1,r) + h_2(x^2,r) + h_3(x^3,r)$. Now the diagonal of the Hessian of $\phi(x,r)$ tell us that

$\frac{\partial^2 h_i(x^i,r)}{\partial x^i x^i} = k(x,r)$, since $\frac{\partial^2 h_i(x^i,r)}{\partial x^i x^i}$ is a function of x^i and r it can be equal to a function of x^j $(i \neq j)$ and r iff k is a function of r alone. Using this fact we can integrate $\frac{\partial^2 h_1(x^1,r)}{\partial x^1 x^1} = k(r)$ to get $\frac{\partial h_1(x^1,r)}{\partial x^1} = k(r)x^1 + g(r)$ and integrating once more $h_1(x^1,r) = \frac{1}{2}k(r)(x^1)^2 + g(r)x^1 + s(r)$. To determine the functions $g(r)$ and $s(r)$ we use the condition $\mathrm{d}_1\phi(r,r) = 0 \Rightarrow \frac{\partial h_1}{\partial x^1}(r^1,r) = 0$ to conclude that $\frac{\partial h_1}{\partial x^1}(r^1,r) = k(r)r^1 + g(r) = 0$ or that $r^1 = -\frac{g(r)}{k(r)}$, this allow us to write h_1 as $h_1(x^1,r) = k(r)\left(\frac{1}{2}(x^1)^2 - r^1x^1 + \frac{s(r)}{k(r)}\right)$. Finally we use the negative-definiteness of the error function once more to get $h_1(r^1,r) = 0$. Since $k(r) \neq 0$ it follows that $\frac{s(r)}{k(r)} = \frac{1}{2}(r^1)^2$.

The error function can then be written in a more compact form as $\phi(x,r) = \frac{1}{2}k(r)(x-r)^T(x-r)$. □

Now we are ready to state the following result.

Proposition 3.2 (Exponential stabilization in Ψ)**.** *For all initial conditions in the open and dense*[2] *set* $\Sigma = \{g \in SE(3) \; : < X_r, X_4(g) >_g \neq 0\}$ *and all the error functions of the form* $\phi(x,r) = \frac{1}{2}k(r)(x-r)^T(x-r)$ *the control law:*

$$
\begin{aligned}
u_1 &= 0 \\
u_2 &= \frac{-\rho_2 \mathrm{d}\psi_2 - < \{0, \ddot{r} - \lambda \frac{d}{dt}(\mathrm{d}_1\phi(x,r))|^T_{x fixed}\}, X_6 >}{< X_r, X_4 >} \qquad \rho_2 > 0 \\
u_3 &= \frac{\rho_3 \mathrm{d}\psi_1 + < \{0, \ddot{r} - \lambda \frac{d}{dt}(\mathrm{d}_1\phi(x,r))|^T_{x fixed}\}, X_5 >}{< X_r, X_4 >} \qquad \rho_3 > 0
\end{aligned}
\tag{25}
$$

exponentially stabilizes the system (7) in the set Ψ.

Proof. Consider the Lyapunov candidate function (23), its time derivative is given by (24), substituting the control law (25) and taking in to account the special form of the error function, one gets:

$$
\frac{d}{dt}V_2 = -\rho_3(\mathrm{d}\psi_1)^2 - \rho_2(\mathrm{d}\psi_2)^2 \tag{26}
$$

which is negative semidefinite. Negative definiteness is proved with an argument similar to the proof of (21), let b^{ψ_1}, b^{ψ_2} be the quadratic lower bounding constants for the functions ψ_1, ψ_2 as defined in (20), respectively. It follows that:

$$
\frac{d}{dt}V_2 \leq -\frac{\rho_3}{b^{\psi_1}}\psi_1 - \frac{\rho_2}{b^{\psi_2}}\psi_2 \tag{27}
$$

To show that trajectories never leave the set Σ it is enough to consider that $\dot{V}_2 \leq 0$, therefore the projection of X_r over X_4 never decreases and thus can never be zero. □

Remarks

The special form of the error function is not necessary to stabilize the system in the set Ψ, however it is very useful since it decouples the attitude control from the position control. It will allow us to chose a control law for u_4 in the next section without disturbing the attitude kinematics. However it reduces the set of possible error functions, forbidding the use of different weights for the error along different directions (one is forced to use $k(r)$ in all directions). This can also be seen as a direct consequence of the reduced set of metrics compatible with the kinematics connection.

Control law (25) forces u_1 to be zero, implying that it is not necessary that the rigid body possesses roll control to stabilize it in Ψ. However roll control can be used if pitch or yaw control fails, constituting a useful redundancy. What is more useful in certain situations is to be able to chose which actuators to use for optimizing fuel consumption or other optimality criteria during the mission, however this approach will not be further addressed in this paper.

[2]The set Σ is open since its complement is the pre-image of the set $\{0\}$ (closed in the usual topology of $\mathbb{R}$) by a continuous (smooth) function. To show that it is dense in $SE(3)$ it suffices to show that its complement is contained in its border. This is trivial attending that the pre-image of any open set in $\mathbb{R}$ containing 0 also contains other points of Σ, by definition of ψ.

Note that control law (25) uses the acceleration of the reference trajectory which is not usual in trajectory tracking. This can be easily explained if one realizes that the attitude control is tracking velocities in trying to align X_r with X_4, therefore since (25) can be viewed as a generalized PD controller it needs acceleration information to accomplish this goal.

Unfortunately control law does not guarantees convergence for all initial conditions, but only for an open and dense set of $SE(3)$, however this is the best that can be achieved since $SE(3)$ is not a simply connected space and this is a topological obstruction to the existence of global stabilizing control laws.

3.5. **Position Control.** Since the orientation of the rigid body is converging to the set Ψ by the action of control inputs u_2 and u_3, it remains to control the forward velocity through control input u_4. The control law for u_4 should be proportional to a measure of the alignment between X_r and X_4, this can trivially be achieved by projecting the reference vector field X_r on X_4, resulting in:

$$u_4 = \frac{< X_r, X_4 >}{< X_4, X_4 >} = \frac{1}{\beta} < X_r, X_4 >= \lambda < X_r, X_4 > \tag{28}$$

Combining (28) with (25) we can asymptotically track the desired reference, this constitutes the main contribution of the paper:

Theorem 3.3 (Asymptotical position tracking)**.** *For all initial conditions in the set Σ and all error functions of the form $\phi(x,r) = \frac{1}{2}k(r)(x-r)^T(x-r)$, control law (25) and (28) makes the system (7) asymptotically track the desired reference $r(t)$.*

In order to prove the result we will need the following standard lemma whose proof can be founded in [14] Appendix A.2.

Lemma 3.4. *Let $f(x) : D \to \mathbb{R}^n, D \subset \mathbb{R}^n$ be a locally Lipschitz vector field on D. If the solution $x(t)$ is bounded and belongs to D for $t \geq 0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover, $x(t) \to L^+$ as $t \to \infty$.*

Proof. The proof will be done in several steps. First we will show that the trajectories of the system are bounded. Next we shall use Lemma 3.4 with Proposition 3.2 to show that the trajectories of the system converge to the largest invariant set in Ψ. Finally it will be shown that the largest invariant set in Ψ is the desired reference $r(t)$.

Boundedness of trajectories

The trajectories of the rotation matrices (living in $SO(3)$) are bounded since $SO(3)$ is a compact space. We only need to show that position of the rigid body is also bounded. We shall use the fact that the trajectories of the system $\dot{x}' = X(x') = \dot{r} - \mathrm{d}_1\phi(x', r)$ are bounded since $\frac{d}{dt}\phi \leq 0$ as shown in (21). Furthermore boundedness of $x'(t)$ implies that there exist non-negative scalars c_1, c_2 and c_3 for each initial condition such that $x'_i(t) < c_i$. The position of the rigid body is governed by:

$$\dot{x} = \lambda < X(x), R \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} > R \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{29}$$

where R is the rotation matrix associated with the group element g. $R\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ is just the first column i of the matrix R, and since R is an orthogonal matrix, the vector i has unitary Euclidean norm. This last fact implies that each element of i verifies $-1 \leq i_i \leq 1$. Having this in mind one can write (29) has:

$$
\begin{aligned}
x(t) &= x(t_0) + \int_{t_0}^{t} \lambda < X(x'(\tau)), i(\tau) > i(\tau) d\tau \\
&= x(t_0) + \begin{bmatrix} \int_{t_0}^{t} X(x'(\tau))^T\, i(\tau)\, i_1(\tau) d\tau \\ \int_{t_0}^{t} X(x'(\tau))^T\, i(\tau)\, i_2(\tau) d\tau \\ \int_{t_0}^{t} X(x'(\tau))^T\, i(\tau)\, i_3(\tau) d\tau \end{bmatrix} \\
&= x(t_0) + \begin{bmatrix} \int_{t_0}^{t} \Big(X_1(x'(\tau)) i_1(\tau) + X_2(x'(\tau)) i_2(\tau) + X_3(x'(\tau)) i_3(\tau) \Big) i_1(\tau)\, d\tau \\ \int_{t_0}^{t} \Big(X_1(x'(\tau)) i_1(\tau) + X_2(x'(\tau)) i_2(\tau) + X_3(x'(\tau)) i_3(\tau) \Big) i_2(\tau)\, d\tau \\ \int_{t_0}^{t} \Big(X_1(x'(\tau)) i_1(\tau) + X_2(x'(\tau)) i_2(\tau) + X_3(x'(\tau)) i_3(\tau) \Big) i_3(\tau)\, d\tau \end{bmatrix} \\
&\leq x(t_0) + \begin{bmatrix} \int_{t_0}^{t} X_1(x'(\tau)) + X_2(x'(\tau)) + X_3(x'(\tau))\, d\tau \\ \int_{t_0}^{t} X_1(x'(\tau)) + X_2(x'(\tau)) + X_3(x'(\tau))\, d\tau \\ \int_{t_0}^{t} X_1(x'(\tau)) + X_2(x'(\tau)) + X_3(x'(\tau))\, d\tau \end{bmatrix} \\
&\leq (c_1 + c_2 + c_3) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad (30)
\end{aligned}
$$

Using a similar argument one can show that trajectories are lower bounded by $\begin{bmatrix} c_2 + c_3 & c_1 + c_3 & c_1 + c_2 \end{bmatrix}^T$.

Convergence to the largest invariant set in Ψ

The system (7) with control laws (25) and (28) is locally Lipschitz since the boundedness assumptions (3.2) on $\phi(x, r)$ and $r(t)$ easily imply that $\frac{\partial \dot{g}}{\partial g}$ is continuous on Σ. Therefore on any compact neighborhood the derivative of $\dot{g}$ with respect to g is bounded, implying local Lipschitz continuity. By applying Lemma 3.4 we conclude that the positive limit set is an invariant set. From (23) we know that trajectories approach Ψ asymptotically, therefore by Lemma 23 they approach the largest invariant set contained in Ψ.

The largest invariant set in Ψ

To study the largest invariant set in Ψ we start by noting that $g \in \Psi \;\Rightarrow\; X_4 = \eta X_r$ for a scalar η and the position kinematics is simplified to $\dot{x} = \dot{r} - \lambda(\mathrm{d}_1\phi(x, r))^T$. Therefore the largest invariant set in Ψ is the desired reference $r(t)$ as shown in (21). □

4. The $SE(2)$ case

The $SE(2)$ case will now be derived as a particular case of the control laws already developed. We will consider that the rigid body is only allowed to move on a horizontal plane, therefore $u_1 = 0$ and $u_2 = 0$. Considering the following base for the Lie algebra $\mathfrak{se}(2)$ of $SE(2)$:

$$
X_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (31)
$$

we get the simplified control law:

$$
\begin{aligned}
u_1 &= \frac{-\rho \mathrm{d}\psi - < \{0, \ddot{r} - \lambda \frac{d}{dt}(\mathrm{d}_1\phi(x, r))^T\}, X_3 >}{< X_r, X_2 >}, \quad \rho > 0 \\
u_2 &= \lambda < X_r, X_2 > \qquad (32)
\end{aligned}
$$

where all the functions are the 2D analogues of the already described ones.

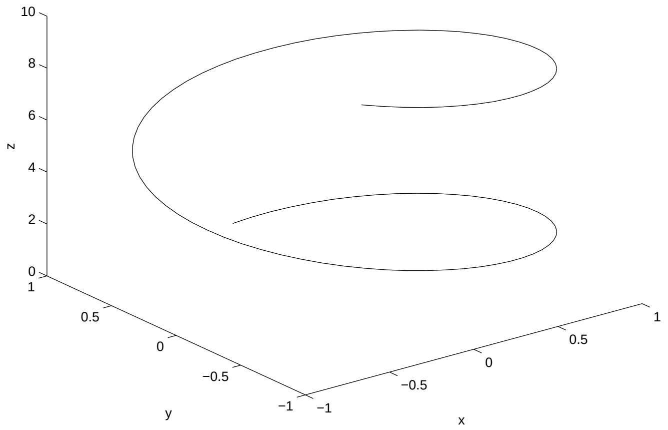

FIGURE 1. Reference trajectory for the $SE(3)$ case $r(t) = \left(cos(\frac{t}{10}), sin(\frac{t}{10}), \frac{t}{10}\right)$, $t \in [0, 100]$.

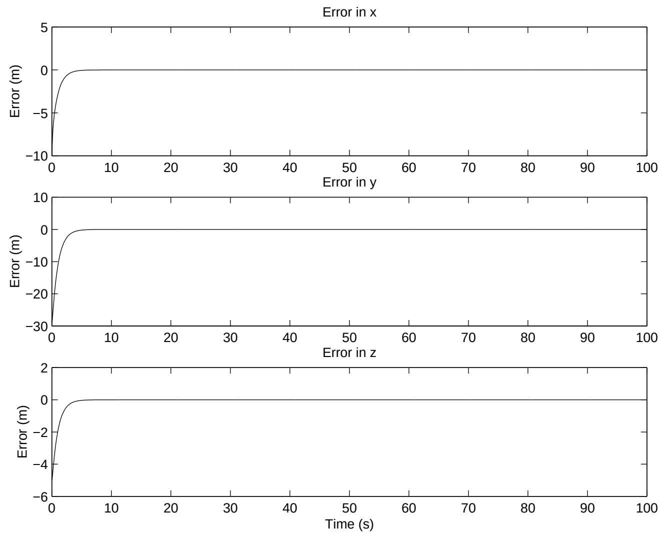

FIGURE 2. Tracking error $(r(t) - x(t))$ for the $SE(3)$ case, initial position $x(0) = (-10, -30, -5)$, initial orientation $R = \mathbf{I}_{3\times3}$.

5. Simulation results

In this section some simulations results are presented for the $SE(3)$ and the $SE(2)$ case. For the $SE(3)$ the used error functions and gains were $\phi(x, r) = \frac{1}{2}(x - r)^T(x - r)$, $\psi_1(a, b) = \psi_2(a, b) = \frac{1}{2}(a - b)^2$, $\rho_1 = 10$ and $\rho_2 = 10$. The metric scalar β was chosen to be unitary. With this values the desired reference was an helix given by $r(t) = \left(\sin(\frac{t}{10}), \cos(\frac{t}{10}), \frac{t}{10}\right)$, $t \in [0, 100]$, represented in figure 1.

The errors between the desired trajectory $r(t)$ and the real trajectory $x(t)$ are represented in figure 2 for an initial position of $x(0) = (-10, -30, -5)$ and an initial orientation of $R = \mathbf{I}_{3\times3}$.

Convergence is very fast and the reference trajectory is tracked with good precision, this motivates the use of more challenging references such as:

$$r(t) = \begin{cases} (t, t) & 0 < t \leq 30 \\ (60 - t, t) & 31 < t \leq 60 \\ (-60 + t, t) & 61 < t \leq 100 \end{cases}$$

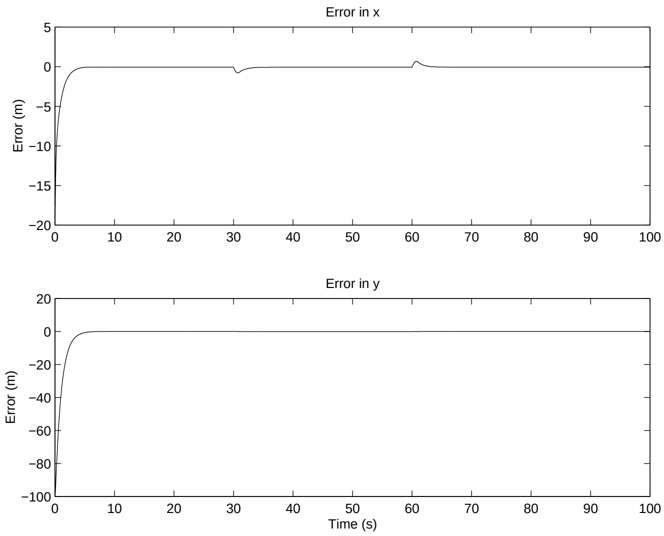

FIGURE 3. Tracking error $(r(t) - x(t))$ for the $SE(2)$ case, initial position $x(0) = (20, 100)$, initial orientation $R = -\mathbf{I}_{2\times 2}$.

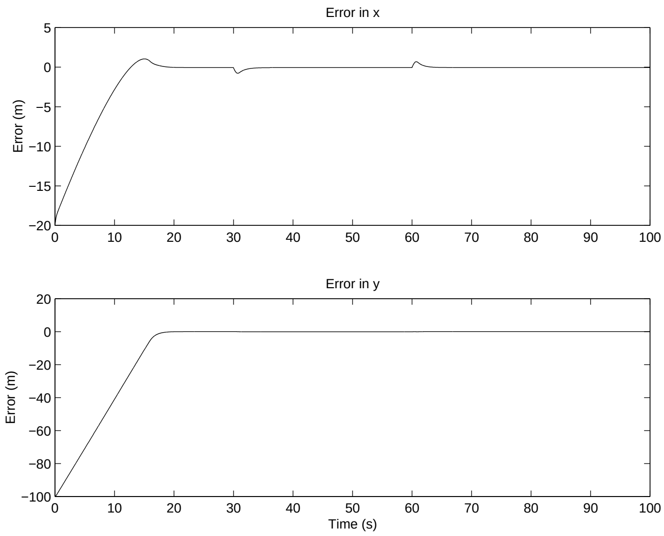

FIGURE 4. Tracking error $(r(t) - x(t))$, for the $SE(2)$ case with bounded actuators, initial position $x(0) = (20, 100)$, initial orientation $R = -\mathbf{I}_{2\times 2}$.

for the $SE(2)$ case. Note that the reference is not twice differentiable violating the conditions of theorem 3.3, this implies that the system will lose track of the reference at the points of non-differentiability as can be seen in figure 3.

Even in this case the results are very impressive since the trajectory is retracked very quickly after being lost. To turn the simulations more realistic the same trajectory was simulated with bounded actuators. The linear and angular velocities are restricted to the set $[-5, 5]$, the results are depicted in figure 4.

The convergence time is much greater and the initial part of the trajectory is not tracked at all as can be seen from figure 5. This was expected since the initial condition is far from the trajectory, however at the points of non-differentiability of the reference the results are very similar with the unrestricted actuators case, evidencing the good performance and robustness for situations not explicitly taken in to account in the theoretical development.

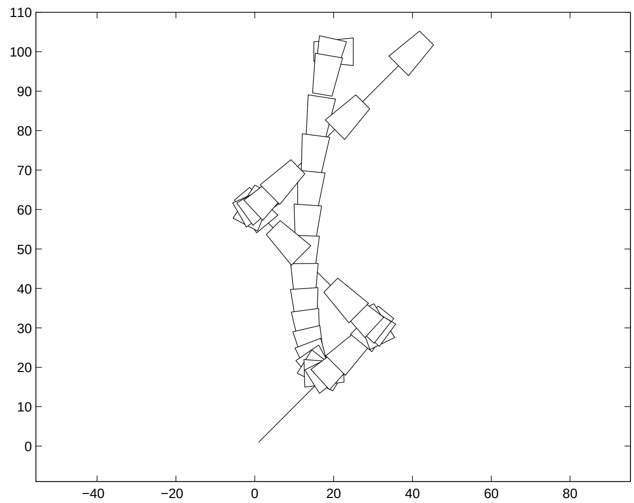

FIGURE 5. Reference trajectory and vehicle location for the $SE(2)$ case with bounded actuators, initial position $x(0) = (20, 100)$, initial orientation $R = -\mathbf{I}_{2\times 2}$.

6. Conclusions

In this paper we have studied the problem of tracking a desired position for an underactuated rigid body in the special euclidean group. It is was shown that the problem is solvable using static state feedback on a open and dense set of $SE(3)$ even if roll control is not possible. The development of the control law was done in a coordinate free way thereby avoiding the unnecessary complications often imposed by possible parameterizations of SE(3). The need to decouple the position motion from the orientation motion led to a reduction of the set of functions measuring the error between the rigid body desired and actual position. This is a direct consequence of the also reduced set of left invariant metrics compatible with the kinematic connection on $SE(3)$. Asymptotic convergence towards the reference trajectory was shown and several simulations were included to shown the algorithm good performance even with non-differentiable reference trajectories.

7. Appendix

7.1. **Time derivative of V.** We shall perform the detailed computations only on the first term of V since the other term is similar.

$$\frac{d}{dt} < X_r, X_5 > = < \frac{D}{dt} X_r, X_5 > + < X_r, \frac{D}{dt} X_5 > \tag{33}$$

Since $X_r = \{0, \dot{r} - \lambda(\mathrm{d}_1\phi(x,r))^T\}$ is already expressed in the inertial frame its covariant derivative is an ordinary time derivative:

$$\frac{D}{dt} X_r = \{0, \ddot{r}\} - \lambda\{0, \frac{d}{dt}(\mathrm{d}_1\phi(x,r))|^T_{xfixed}\} - \lambda\{0, \frac{d}{dt}(\mathrm{d}_1\phi(x,r))|^T_{rfixed}\} \tag{34}$$

The last term can be written as:

$$\lambda\{0, \frac{d}{dt}(\mathrm{d}_1\phi(x,r))|^T_{rfixed}\} = \lambda u_4\{0, \frac{\partial^2\phi(x,r)}{\partial x^i x^j} X_4\} \tag{35}$$

where $\frac{\partial^2\phi(x,r)}{\partial x^i x^j}$ represents the Hessian of $\phi(x,r)$.

Finally the inner product can be computed, recollecting (34) and (35) we get:

$$< \frac{D}{dt} X_r, X_5 > = < \{0, \ddot{r} - \lambda \frac{d}{dt}(\mathrm{d}_1 \phi(x,r))|^T_{x fixed}\}, X_5 > - \lambda u_4 < \frac{\partial^2 \phi(x,r)}{\partial x^i x^j} X_4, X_5 > \tag{36}$$

by the same reasoning $< \frac{D}{dt} X_r, X_6 >$ is:

$$< \frac{D}{dt} X_r, X_6 > = < \{0, \ddot{r} - \lambda \frac{d}{dt}(\mathrm{d}_1 \phi(x,r))|^T_{x fixed}\}, X_6 > - \lambda u_4 < \frac{\partial^2 \phi(x,r)}{\partial x^i x^j} X_4, X_6 > \tag{37}$$

The second term of (33) is given by:

$$\begin{aligned} < X_r, \frac{D}{dt} X_5 > &= < X_r, \nabla_{\dot{g}} X_5 > \\ &= < X_r, \nabla_{g(X_i u_i)} X_5 > \\ &= u_i < X_r, \nabla_{X_i(g)} X_5 > \end{aligned} \tag{38}$$

which by definition is given by:

$$\begin{aligned} < X_r, \frac{D}{dt} X_5 > &= u_1 < X_r, \Gamma^k_{15} X_k > + u_2 < X_r, \Gamma^k_{25} X_k > \\ &\quad + u_3 < X_r, \Gamma^k_{35} X_k > + u_4 < X_r, \Gamma^k_{45} X_k > \\ &= u_1 < X_r, X_6 > - u_3 < X_r, X_4 > \end{aligned} \tag{39}$$

by the same reasoning $< X_r, \frac{D}{dt} X_6 >$ is given by:

$$< X_r, \frac{D}{dt} X_6 > = u_2 < X_r, X_4 > - u_1 < X_r, X_5 > \tag{40}$$

Finally we get:

$$\begin{aligned} \frac{d}{dt} < X_r, X_5 > &= < \{0, \ddot{r} - \lambda \frac{d}{dt}(\mathrm{d}_1 \phi(x,r))|^T_{x fixed} \cdot \dot{r}\}, X_5 > - \lambda u_4 < \frac{\partial^2 \phi(x,r)}{\partial x^i x^j} X_4, X_5 > \\ &\quad + u_1 < X_r, X_6 > - u_3 < X_r, X_4 > \end{aligned} \tag{41}$$

and

$$\begin{aligned} \frac{d}{dt} < X_r, X_5 > &= < \{0, \ddot{r} - \lambda \frac{d}{dt}(\mathrm{d}_1 \phi(x,r))|^T_{x fixed} \cdot \dot{r}\}, X_6 > - \lambda u_4 < \frac{\partial^2 \phi(x,r)}{\partial x^i x^j} X_4, X_6 > \\ &\quad + u_2 < X_r, X_4 > - u_1 < X_r, X_5 > \end{aligned} \tag{42}$$

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