$L^2(\mathbb{R})$ Nonstationary Processes and the Sampling Theorem

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***Abstract*—In [1], a sampling theorem for nonstationary random processes is developed, under the condition that the two-dimensional (2-D) power spectrum (2DPS) of the process has compact support. In this letter, it is shown that, for $L^2(\mathbb{R})$ processes, only a one-dimensional (1-D) restriction on the marginal along time of the time-frequency distribution is necessary to guarantee the compactness of the 2DPS in the 2-D plane. As a direct consequence, it is observed that under mild conditions, a nonstationary autocorrelation function of a bandpass $L^2(\mathbb{R})$ process is nearly stationary in small time intervals. The influence of this result in real-time detection of nonstationary stochastic signals is discussed.**

***Index Terms*—$L^2(\mathbb{R})$ nonstationary processes, real-time detection, sampling theorem, time-frequency distribution.**

I. Introduction

The SAMPLING theorem has been extensively referred to in signal and image processing literature, either in the deterministic case, (one-dimensional (1-D) [2] and two-dimensional (2-D) [3]), as well as in wide sense stationary (WSS) stochastic processes [2]. In [1], it is shown that a nonstationary process can be recovered from its samples if its 2-D power spectrum (2DPS) has compact support in $\mathbb{R}^2$.

This letter reports on the sampling of $L^2(\mathbb{R})$ random processes and proves, as a main result, that a 1-D compactness restraint on the marginal along time of the time-frequency distribution (MTTFD) is equivalent to the 2-D restriction on the 2DPS, consisting thus on a sufficient condition to perform, in the mean-square sense, sampling of nonstationary bandlimited random processes.

It is also shown that the 1-D compactness restraint on the MTTFD is equivalent to impose that the time-frequency distribution (TFD) has, for all time values, compact support in the frequency domain. This result is the nonstationary equivalent of the definition of a stationary bandlimited process. The TFD can thus be viewed as an extension of the stationary power spectrum for nonstationary processes.

Defining the autocorrelation function of a bandpass $L^2(\mathbb{R})$ process through time and time-lag variables, we conclude that, under some mild conditions, the variation along time is slower than along time-lag. This result confirms that a nonstationary autocorrelation function is nearly stationary in small time intervals, and is used in [4] to adapt compact support wavelets to Gaussian transient processes decomposition in detection problems. Furthermore, in real-time detection of nonstationary signals, this result shows that the rate at which the sequential quadratic tests must be performed may be slower than the Nyquist rate, thus reducing the computational complexity of the processor.

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II. One-Dimensional (1-D) Condition for Sampling Nonstationary Processes

Let $s(t)$, $t \in \mathbb{R}$, be a zero-mean $L^2(\mathbb{R})$ nonstationary stochastic process with autocorrelation function $k_s(t_1, t_2)$ and 2DPS $K_s(\omega_1, \omega_2) = FT_{t_1}\,[FT_{t_2}[k_s(t_1, t_2)](-\omega_2)](\omega_1)$, where $FT[\cdot](\omega)$ denotes the Fourier transform (FT). The autocorrelation function $k_s(t_1, t_2)$ is positive semidefinite by definition [5], i.e., for any sequence $t_1, t_2, \cdots, t_n$ and any complex constants $\alpha_1, \alpha_2, \cdots, \alpha_n$, one has

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i^* \alpha_j k_s(t_i, t_j) \geq 0. \tag{1}$$

Suppose $k_s(t_1, t_2)$ is continuous. Since $k_s(t_1, t_2) \in L^2(\mathbb{R}^2)$, condition (1) is equivalent to

$$\iint_{-\infty}^{\infty} y^*(t_1) k_s(t_1, t_2) y(t_2)\, dt_1\, dt_2 \geq 0, \quad \forall\, y(t) \in L^2(\mathbb{R}). \tag{2}$$

Assuming that $k_s(t, t) \in L^1(\mathbb{R})$, it may be shown that the autocorrelation function is related to its eigenfunctions $\phi_i(t)$ and eigenvalues λ_i by a Mercer-like expansion [6]

$$k_s(t_1, t_2) = \sum_{i=1}^{\infty} \lambda_i \phi_i(t_1) \phi_i^*(t_2). \tag{3}$$

In general, the literature presents Mercer's Theorem in a stationary signal context, t_1 and t_2 being defined in finite time intervals. However, as pointed out in [6], when the signals belong to $L^2(\mathbb{R})$ and $k_s(t, t) \in L^1(\mathbb{R})$, the corresponding time intervals are extensible to the entire real line.

Next, we show that the 2DPS $K_s(\omega_1, \omega_2)$ is also a positive semidefinite function. We can write the Fourier equivalent of (3)

$$K_s(\omega_1, \omega_2) = \sum_{i=1}^{\infty} \lambda_i \Phi_i(\omega_1) \Phi_i^*(\omega_2). \tag{4}$$

Applying the Parseval relation to (2) and using (3) and (4), one gets $\forall Y(\omega) \in L^2(\mathbb{R})$

$$\iint_{-\infty}^{\infty} y^*(t_1)k_s(t_1, t_2)y(t_2)\,dt_1\,dt_2 = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} Y^*(\omega_1)K_s(\omega_1, \omega_2)Y(\omega_2)\,d\omega_1\,d\omega_2 \geq 0 \tag{5}$$

showing that $K_s(\omega_1, \omega_2)$ is positive semidefinite. Thus, the Schwarz inequality translates to

$$|K_s(\omega_1, \omega_2)|^2 \leq K_s(\omega_1, \omega_1)K_s(\omega_2, \omega_2). \tag{6}$$

We must remark that although in general, $K_s(\omega_1, \omega_2)$ is complex (as is also $k_s(t_1, t_2)$), it can be easily verified that $K_s(\omega, \omega)$ is real and positive [see in the sequel expression (12), where $\lambda_i \geq 0, \forall i$, due to the positive-semidefiniteness of $k_s(t_1, t_2)$]. All the relevant mathematical results are formally proved elsewhere [6].

The sampling theorem for nonstationary random processes presented in [1] is established assuming that $s(t)$ is a bandpass process, that is, its 2DPS is characterized by

$$K_s(\omega_1, \omega_2) = 0 \quad \text{for} \quad |\omega_1| > a \quad \text{and for} \quad |\omega_2| > a. \tag{7}$$

Under the above condition, a signal $s(t)$ can be recovered, in the mean square sense, from its samples, i.e.

$$E\left[\left\{s(t) - \sum_{n=-\infty}^{\infty} s(nT_s)\,\mathrm{sinc}(t/T_s - n)\right\}^2\right] = 0, \quad \forall t \in \mathbb{R} \tag{8}$$

when the sampling interval is such that $T_s < \pi/a$. The sinc functions correspond to the decomposition basis, and the samples $s(nT_s)$ are the resulting coefficients. However, condition (7) is equivalent to

$$K_s(\omega, \omega) = 0 \quad \text{for} \quad |\omega| > a. \tag{9}$$

From (6), if $K_s(\omega, \omega)$ satisfies (9), then (7) is also verified. On the contrary, (9) results immediately from taking $\omega_1 = \omega_2 = \omega$ in (7).

III. Time-Frequency Distribution

Defining the time-frequency distribution (TFD) and the marginal along time of the TFD (MTTFD), respectively, by

$$S_s(t, \omega) = FT_\tau\left[k_s\left(t + \frac{\tau}{2}, t - \frac{\tau}{2}\right)\right](\omega)$$

and

$$P_s(\omega) = \int_{-\infty}^{\infty} S_s(t, \omega)\,dt \tag{10}$$

then, using Mercer's Theorem and the FT definition, it is straightforward to show that

$$\hat{S}_s(\Omega, \omega) = FT_t[S_s(t, \omega)](\Omega) = K_s\left(\omega + \frac{\Omega}{2}, \omega - \frac{\Omega}{2}\right) \tag{11}$$

and

$$P_s(\omega) = \sum_{i=1}^{\infty} \lambda_i |\Phi_i(\omega)|^2 = K_s(\omega, \omega). \tag{12}$$

Thus, $K_s(\omega, \omega)$ is in fact the MTTFD. Regarding the conditions for the sampling theorem to hold, it is necessary that, for $|\omega| > a$

$$P_s(\omega) = K_s(\omega, \omega) = 0 \quad \Leftrightarrow \quad \hat{S}_s(\Omega, \omega) = 0, \quad \forall \Omega \quad \Leftrightarrow \quad S_s(t, \omega) = 0, \qquad \forall t. \tag{13}$$

These results follow directly from the application of the Schwarz inequality. Clearly, the MTTFD corresponds to the signal's energy distribution along the frequency axis.

When $s(t)$ is a real process, some authors (see [7]) note that, due to the fact that the TFD can take negative values in some situations, it lacks physical meaning to represent a time varying equivalent of the stationary power spectrum. However, the TFD is closely related to the Wigner transform [8], considered to be a convenient tool to analyze signals with time-varying frequency components, like chirps. The TFD negative values appear in signals with multiple time or frequency maxima, and correspond to cross-terms with no physical meaning. In the present context, the Schwarz inequality in (6) shows that if the TFD has compact support in the frequency domain for all time values, then it is possible to recover with null mean-squared error (MSE) the original signal from its samples. Neglecting the effect of the cross-terms, the TFD can thus be considered as a time-varying equivalent of the stationary power spectrum, also from a sampling theory point of view.

IV. Influence of the Schwarz Inequality in Detection Problems

In this section, we consider that $s(t)$ belongs to a particular class of bandpass nonstationary signals with most of their energy lying in the interval $I = [-\omega_{\max}; -\omega_{\min}] \cup [\omega_{\min}; \omega_{\max}]$, i.e., $S_s(t, \omega) \simeq 0$ for $\omega \notin I, \forall t \in \mathbb{R}$. Let $\Delta\omega = \omega_{\max} - \omega_{\min}$ and $L_\omega(\Omega) = \hat{S}_s(\Omega, \omega)$. Then $\forall \omega \in I$ and $|\Omega| > \Delta\omega$, and we have

$$L_\omega(0) = P_s(\omega)$$

and

$$|L_\omega(\Omega)|^2 \leq P_s\left(\omega + \frac{\Omega}{2}\right) P_s\left(\omega - \frac{\Omega}{2}\right) \simeq 0. \tag{14}$$

Thus, while $s(t)$ is a bandpass process with frequency components represented by $S_s(t, \omega)$, the behavior of $k_s(t + \tau/2, t - \tau/2)$ in order with the time variable t, represented in the frequency domain by the function $L_\omega(\Omega)$ ($\omega \in I$), is lowpass.

This result is of interest in problems related to the real-time detection of nonstationary bandpass signals. Two aspects are highlighted here.

First, we consider the decomposition of a nonstationary signal in a small number of coefficients to reduce the computational complexity of the evaluation of the likelihood ratio. As shown in [4], a convenient decomposition corresponds to the wavelet transform using a mother wavelet of compact support, which is efficient for real-time applications. The choice of the mother wavelet is performed in the frequency domain, and a strong simplification in the optimization problem results from assuming that the frequency contents of the nonstationary process do not change in the support of the wavelets. Due to the fact that an autocorrelation function is positive semidefinite and the Schwarz inequality holds, the result presented in this section shows that the variation along time of the second order statistics of this class of nonstationary processes is lowpass and therefore, the assumption of "near-stationarity" in small time intervals is valid.

Another relevant problem in real-time detection consists in the choice of the rate at which the likelihood test (LT) must be evaluated. It is assumed, in this case, that the continuous-time observation process is sampled and a LT is performed from a window consisting of the last N discrete-time samples. As time goes on and new samples arrive, the window is shifted and the process is repeated on and on. Clearly, if the LTs are computed at every sampling interval, the shift error between the observation process when a signal is present and the model, represented by the covariance matrix, is small. However, using an LT rate equal to the sampling rate leads to a huge computational complexity of the processor. The choice of the LT rate must not be performed according to the frequency components of the signal (which are obviously related to the sampling rate), but considering the evolution along time of the statistics of the process. In the limit, for stationary processes, the statistics are constant and any shift in the observation window statistically returns the same result. In the case of second-order nonstationary bandpass processes, the maximum allowable shift corresponds to the time interval where the autocorrelation function remains approximately stationary. The Schwarz inequality in (14) shows that, for many processes, the LT rate can be significantly slower than the sampling rate.

V. Conclusions

This letter reports on the sampling of nonstationary $L^2(\mathbb{R})$ random processes. It is shown herein that only a simple 1-D compactness restraint on the marginal along time of the time-varying power spectrum is necessary for the sampling of such processes. Furthermore, it strengthens the known idea that the time-frequency distribution is an extension of the stationary power spectrum for nonstationary processes. At last, it is stated that, for a particular class of bandpass processes, the time variation of the autocorrelation function is slow comparing to the time lag's, and some insight related to real-time detection of nonstationary processes is presented.

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