Landmark Based Nonlinear Observer for Rigid Body Attitude and Position Estimation

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***Abstract*— This work proposes a nonlinear observer for position and attitude estimation on $\mathrm{SE}(3)$. Using a Lyapunov function based on the landmark measurement error, almost global exponential stability (GES) of the desired attitude and position equilibrium points is obtained. It is shown that the derived feedback law is an explicit function of the landmark measurements and velocity readings, and that the landmark geometry characterizes the asymptotic convergence of the closed loop system solution. Almost global exponential stabilization in the presence of biased velocity readings is also achieved. Simulation results for trajectories described by time-varying linear and angular velocities and for distinct initial conditions on $\mathrm{SE}(3)$ are presented to illustrate the stability and convergence properties of the observer.**

I. INTRODUCTION

The classical problem of attitude and position estimation is often subject to new advances and enriching insights, despite its extensive historical background. Research on the problem of deriving a stabilizing law for systems evolving on non-Euclidean spaces, namely $\mathrm{SO}(3)$ and $\mathrm{SE}(3)$, can be found in [3], [5], [7], [9], [13], where the discussion on the topological characteristics and limitations for achieving global stabilization on the $\mathrm{SO}(3)$ manifold provide important guidelines for the design of observers.

Recently, nonlinear attitude observers on $\mathrm{SO}(3)$ have been proposed. In [12] a locally exponentially convergent attitude observer is proposed using a monocular camera and inertial sensors, and observability conditions are studied. An eventually globally exponentially convergent angular velocity observer, expressed in the Euler quaternion representation, is derived in [13] by exploiting attitude and torque measurements. The nonlinear attitude observer proposed in [14] also yields globally exponential convergence to the origin, using attitude and inertial measurements. A nonlinear complementary attitude filter is derived in [8], using the rotation matrix parametrization and producing an almost globally exponentially convergent estimator.

However, [8], [13], [14] assume that an explicit quaternion/rotation matrix attitude measurement is available, obtained by batch processing sensor measurements such as landmarks, image based features, and vector readings. This approach does not take into account the sensor characteristics, and the impact of the sensed quantities geometry on the estimation problem cannot be analyzed. Also, position estimation is not addressed in these references.

In this work, the position and attitude of a rigid body are estimated by exploiting landmark readings and velocity measurements directly. Based on a conveniently defined landmark estimation error, a Lyapunov function is proposed and a constructive derivation of the feedback law is presented. This framework provides a geometric insight on the necessary and sufficient conditions for attitude and position determination.

The derived output feedback architecture yields almost global exponential stability (GES) of the desired equilibrium point on $\mathrm{SE}(3)$. Almost global exponential stability is extended for the case of biased linear velocity readings. The influence of the landmark configuration on the directions of the closed-loop trajectories is also analyzed.

As pointed out in [7], [9], a topological obstacle to continuous global stabilization arises from the fact that $\mathrm{SE}(3)$ is not diffeomorphic to an Euclidean vector space, which implies that there is no continuous state feedback law that yields global asymptotic stability of an equilibrium point. The relaxation from global to almost global stability adopted in this paper provides a suitable framework for $\mathrm{SE}(3)$, where convergence to the equilibrium point is guaranteed for any initial condition outside a set of zero measure.

The paper is organized as follows. In Section II, the position and attitude estimation problem is introduced and the available sensor information is detailed. The attitude and position observer is derived in Section III by crafting a Lyapunov function based on the landmark measurement error, and necessary and sufficient conditions for attitude and position determination are discussed. The proposed Lyapunov function is decoupled into independent position and attitude components that are addressed separately, yielding almost global exponentially stabilizing feedback laws. The presence of unknown velocity sensor bias is also studied. In Section IV the feedback law is expressed as a function of the sensor readings (output feedback architecture) and it is shown that the asymptotic convergence of the system trajectories is determined by the landmark geometry. The simulation results of Section V illustrate the observer properties for time-varying linear and angular velocities. Section VI presents concluding remarks and comments on future work.

II. PROBLEM FORMULATION

Landmark based navigation, illustrated in Fig. 1, can be summarized as the problem of determining attitude and posi-

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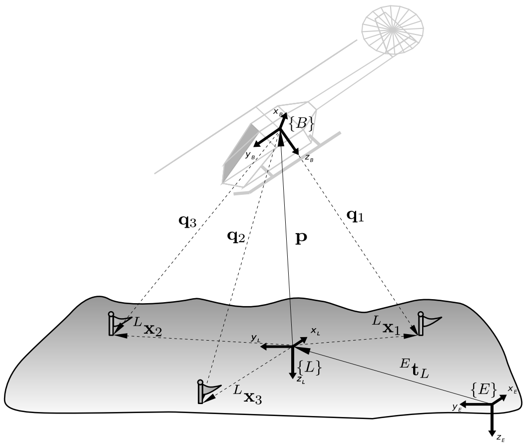


Fig. 1. Landmark Based Navigation

tion of a rigid body using landmark observations and velocity measurements. The rigid body kinematics are described by

$$\dot{\bar{\mathcal{R}}} = \bar{\mathcal{R}}\left[\bar{\boldsymbol{\omega}}\times\right], \qquad \dot{\bar{\mathbf{p}}} = \bar{\mathbf{v}} - \left[\bar{\boldsymbol{\omega}}\times\right]\bar{\mathbf{p}},$$

where $\bar{\mathcal{R}}$ is the shorthand notation for the rotation matrix ${}^{L}_{B}\mathbf{R}$ from body frame $\{B\}$ to local frame $\{L\}$ coordinates, $\bar{\boldsymbol{\omega}}$ and $\bar{\mathbf{v}}$ are the body angular and linear velocities, respectively, expressed in $\{B\}$, $\bar{\mathbf{p}}$ is the position of the rigid body with respect to $\{L\}$ expressed in $\{B\}$, and $[\mathbf{a}\times]$ is the skew symmetric matrix defined by the vector $\mathbf{a} \in \mathbb{R}^3$ such that $[\mathbf{a}\times]\,\mathbf{b} = \mathbf{a} \times \mathbf{b}$, $\mathbf{b} \in \mathbb{R}^3$. Without loss of generality, the local frame is defined by translating the Earth frame $\{E\}$ to the landmarks centroid, as depicted in Fig. 1.

The body linear and angular velocities are measured by a Doppler sensor and a rate gyro sensor triad, respectively

$$\boldsymbol{\omega}_r = \bar{\boldsymbol{\omega}}, \quad \mathbf{v}_r = \bar{\mathbf{v}}. \tag{1}$$

The landmark measurements, illustrated in Fig. 1, are obtained by on-board sensors that are able to track terrain characteristics, such as CCD cameras or ladars,

$$\mathbf{q}_{r\,i} = \bar{\mathbf{q}}_i = \bar{\mathcal{R}}'{}^{L}\bar{\mathbf{x}}_i - \bar{\mathbf{p}}, \tag{2}$$

where ${}^{L}\bar{\mathbf{x}}_i$ represent the coordinates of landmark i in the local frame $\{L\}$. By the definition of the frame $\{L\}$ origin, the coordinates of the landmarks centroid are invariant with respect to the frame $\{B\}$ orientation, that is,

$$\sum_{i=1}^{n} {}^{L}\bar{\mathbf{x}}_i = 0 \Rightarrow \sum_{i=1}^{n} \mathcal{R}'{}^{L}\bar{\mathbf{x}}_i = 0. \tag{3}$$

The proposed observer which estimates the position and attitude of the rigid body takes the form

$$\dot{\hat{\mathcal{R}}} = \hat{\mathcal{R}}\left[\hat{\boldsymbol{\omega}}\times\right], \quad \dot{\hat{\mathbf{p}}} = \hat{\mathbf{v}} - \left[\hat{\boldsymbol{\omega}}\times\right]\hat{\mathbf{p}},$$

where $\hat{\boldsymbol{\omega}}$ and $\hat{\mathbf{v}}$ are the estimated body angular and linear velocity, respectively.

The position and attitude errors are defined as $\tilde{\mathbf{p}} := \hat{\mathbf{p}} - \bar{\mathbf{p}}$ and $\tilde{\mathcal{R}} := \hat{\mathcal{R}}\bar{\mathcal{R}}'$, respectively. The Euler angle-axis parametrization of the rotation error matrix $\tilde{\mathcal{R}}$ is described by the rotation vector $\boldsymbol{\lambda} \in \mathrm{S}(2)$ and by the rotation angle $\theta \in \begin{bmatrix}0 & \pi\end{bmatrix}$, yielding the DCM formulation [11] $\tilde{\mathcal{R}} = \mathrm{rot}(\theta, \boldsymbol{\lambda}) := \cos(\theta)\mathbf{I} + \sin(\theta)\left[\boldsymbol{\lambda}\times\right] + (1-\cos(\theta))\boldsymbol{\lambda}\boldsymbol{\lambda}'$. The attitude and position error dynamics are a function of the linear and angular velocity estimates and given by

$$\dot{\tilde{\mathcal{R}}} = \tilde{\mathcal{R}}\left[\bar{\mathcal{R}}(\hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}})\times\right], \tag{4}$$

$$\dot{\tilde{\mathbf{p}}} = (\hat{\mathbf{v}} - \bar{\mathbf{v}}) - \left[\bar{\boldsymbol{\omega}}\times\right]\tilde{\mathbf{p}} + \left[\hat{\mathbf{p}}\times\right](\hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}}). \tag{5}$$

The attitude and position feedback laws are obtained by defining $\hat{\boldsymbol{\omega}}$ and $\hat{\mathbf{v}}$ as a function of the velocity readings (1) and landmark observations (2), so that the closed loop position and attitude estimation errors converge asymptotically to the origin, i.e. $\tilde{\mathcal{R}} \to \mathbf{I}, \tilde{\mathbf{p}} \to 0$ as $t \to \infty$.

III. OBSERVER SYNTHESIS

The proposed candidate Lyapunov function is characterized by the estimation error of the landmark readings

$$V = \frac{1}{2}\sum_{i=1}^{n} \|\hat{\mathbf{q}}_i - \mathbf{q}_{r\,i}\|^2. \tag{6}$$

where $\hat{\mathbf{q}}_i := \hat{\mathcal{R}}'{}^{L}\bar{\mathbf{x}}_i - \hat{\mathbf{p}}$ denotes the coordinates of landmark i computed from the observer estimates $\hat{\mathcal{R}}$ and $\hat{\mathbf{p}}$, and the known vector ${}^{L}\bar{\mathbf{x}}_i$. The landmark geometry is considered to yield sufficient information about attitude and position if the Lyapunov global minimum at $\hat{\mathbf{q}}_i = \mathbf{q}_{r\,i}$, $i = 1..n$ is equivalent to $\tilde{\mathcal{R}} = \mathbf{I}$, $\tilde{\mathbf{p}} = 0$, i.e. the landmark measurements in the body frame correspond to a single attitude and position configuration.

In this section, the attitude and position feedback laws are formulated. It is shown that the Lyapunov function is a linear combination of two independent position and attitude Lyapunov functions, which allows for the separate derivation of the position and attitude feedback laws. The closed loop system is demonstrated to have an almost GES equilibrium point $\tilde{\mathcal{R}} = \mathbf{I}$, $\tilde{\mathbf{p}} = 0$. The compensation of velocity sensor bias is also addressed.

A. Lyapunov Function Properties

The proposed Lyapunov function can be described as a linear combination of distinct position and attitude components, $V = V_{\mathcal{R}} + V_p$, given by

$$\begin{aligned} V_{\mathcal{R}} &= \frac{1}{2}\sum_{i=1}^{n}\|\hat{\mathbf{x}}_i - \bar{\mathbf{x}}_i\|^2 = \mathrm{tr}\left[(\mathbf{I} - \tilde{\mathcal{R}})\mathbf{X}\mathbf{X}'\right] \\ &= (1-\cos(\theta))\,\boldsymbol{\lambda}'\mathbf{P}\boldsymbol{\lambda}, \quad V_p = \frac{n}{2}\tilde{\mathbf{p}}'\tilde{\mathbf{p}}, \end{aligned} \tag{7}$$

where $\hat{\mathbf{x}}_i := \hat{\mathcal{R}}'{}^{L}\bar{\mathbf{x}}_i$, $\bar{\mathbf{x}}_i := \bar{\mathcal{R}}'{}^{L}\bar{\mathbf{x}}_i$, $\mathbf{X} := \begin{bmatrix}{}^{L}\bar{\mathbf{x}}_1 & \cdots & {}^{L}\bar{\mathbf{x}}_n\end{bmatrix} \in \mathrm{M}(3,n)$ and $\mathbf{P} := \mathrm{tr}(\mathbf{X}\mathbf{X}')\mathbf{I} - \mathbf{X}\mathbf{X}' \in \mathrm{M}(3)$. The time derivatives of proposed Lyapunov functions are expressed by

$$\begin{aligned} \dot{V}_{\mathcal{R}} &= \left[\mathbf{X}\mathbf{X}'\tilde{\mathcal{R}} - \tilde{\mathcal{R}}'\mathbf{X}\mathbf{X}'\otimes\right]'\bar{\mathcal{R}}(\hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}}) \\ &= \boldsymbol{\lambda}'\mathbf{P}\mathbf{Q}(\theta,\boldsymbol{\lambda})\bar{\mathcal{R}}(\hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}}), \end{aligned} \tag{8}$$

$$\dot{V}_p = n\tilde{\mathbf{p}}'(\left[\hat{\mathbf{p}}\times\right](\hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}}) + (\hat{\mathbf{v}} - \bar{\mathbf{v}})), \tag{9}$$

where $\mathbf{Q}(\theta,\boldsymbol{\lambda}) = \sin(\theta)\mathbf{I} + (1-\cos(\theta))\left[\boldsymbol{\lambda}\times\right]$ and $\otimes$ is the unskew operator such that $[[\mathbf{a}\times]\otimes] = \mathbf{a}, \mathbf{a} \in \mathbb{R}^3$.

The decoupling property of the Lyapunov function allows for the attitude and position estimation problems to be addressed separately [4]. The feedback law for the attitude dynamics (4) is derived using the Lyapunov function $V_{\mathcal{R}}$, while the position feedback law is obtained by studying the position error (5) using V_p.

B. Attitude Feedback Law

In this section, the feedback law for the estimation of attitude is derived. To that end, the conditions which guarantee that $\|{}^B\hat{\mathbf{x}}_i - {}^B\bar{\mathbf{x}}_i\| = 0$ if and only if $\tilde{\mathcal{R}} = \mathbf{I}$ are presented. The geometric placement of the landmarks is required to satisfy the following assumption:

Assumption 1 (Landmark Readings): There are at least three noncolinear landmarks.

From (3), the condition of Assumption 1 can be expressed as $\exists_{i\neq j}\forall_{\alpha\in\mathbb{R}} : {}^L\bar{\mathbf{x}}_i \neq \alpha {}^L\bar{\mathbf{x}}_j$ or $\mathrm{rank}(\mathbf{X}) \geq 2$. As shown in the next lemma, Assumption 1 is equivalent to $V_{\mathcal{R}} > 0$.

Lemma 1: The Lyapunov function $V_{\mathcal{R}}$ has a unique global minimum if and only if Assumption 1 is verified

$$\forall_{\tilde{\mathcal{R}}\neq\mathbf{I}} V_{\mathcal{R}} > 0 \quad \text{if and only if} \quad \exists_{i\neq j}\forall_{\alpha\in\mathbb{R}} : {}^L\bar{\mathbf{x}}_i \neq \alpha {}^L\bar{\mathbf{x}}_j.$$

Proof: See Appendix A. ■

It is instructive to analyze why $\mathrm{rank}(\mathbf{X}) = 1$ is not sufficient to determine attitude. If $\forall_{i,j} \frac{{}^L\mathbf{x}_i}{\|{}^L\mathbf{x}_i\|} = \pm\frac{{}^L\mathbf{x}_j}{\|{}^L\mathbf{x}_j\|}$, simple algebraic manipulations show that any attitude error $\tilde{\mathcal{R}} = \mathrm{rot}(\theta, \boldsymbol{\lambda})$ such that $\boldsymbol{\lambda} = \frac{{}^L\mathbf{x}_i}{\|{}^L\mathbf{x}_i\|}$ implies $V_{\mathcal{R}} = 0$. This is related to the well known fact that a single vector observation (such as the Earth's magnetic field) yields attitude information except for the rotation about the vector itself [15], that is, $\boldsymbol{\lambda} = \frac{{}^L\mathbf{x}_i}{\|{}^L\mathbf{x}_i\|}$.

Given the Lyapunov function derivatives along the system trajectories (8), a feedback law is defined to drive the attitude error to zero,

$$\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}_r - K_\omega \mathbf{s}_\omega, \tag{10}$$

where the feedback term is given by

$$\mathbf{s}_\omega := \bar{\mathcal{R}}' \left[\mathbf{X}\mathbf{X}'\tilde{\mathcal{R}} - \tilde{\mathcal{R}}'\mathbf{X}\mathbf{X}'\otimes\right] = \bar{\mathcal{R}}'\mathbf{Q}'(\theta,\boldsymbol{\lambda})\mathbf{P}\boldsymbol{\lambda}, \tag{11}$$

and K_ω is a positive scalar. The attitude feedback yields an autonomous closed loop attitude dynamics

$$\dot{\tilde{\mathcal{R}}} = K_\omega(\mathbf{X}\mathbf{X}' - \tilde{\mathcal{R}}\mathbf{X}\mathbf{X}'\tilde{\mathcal{R}}), \tag{12}$$

and a negative semidefinite derivative for $V_{\mathcal{R}}$ given by $\dot{V}_{\mathcal{R}} = -K_\omega \mathbf{s}_\omega' \mathbf{s}_\omega \leq 0$. It is immediate that the attitude feedback law produces a Lyapunov function that decreases along the system trajectories and, by LaSalle's invariance principle, guarantees global convergence to the largest invariant set contained in the set defined by $\dot{V}_{\mathcal{R}} = 0$.

Lemma 2: Under Assumption 1, the set of points where $\dot{V}_{\mathcal{R}} = 0$ is given by

$$\begin{aligned} C_{V_{\mathcal{R}}} &= \{\tilde{\mathcal{R}} \in \mathrm{SO}(3) : \tilde{\mathcal{R}} = \mathbf{I} \vee \tilde{\mathcal{R}} = \mathrm{rot}(\pi, \boldsymbol{\lambda} \in \mathrm{eigvec}(\mathbf{P}))\} \\ &= \{(\theta,\boldsymbol{\lambda}) \in \mathrm{D}_\lambda : \theta = 0 \vee (\theta = \pi, \boldsymbol{\lambda} \in \mathrm{eigvec}(\mathbf{P}))\}, \end{aligned}$$

where $\mathrm{D}_\lambda = \begin{bmatrix} 0 & \pi \end{bmatrix} \times \mathrm{S}(2)$.

Proof: See Appendix A. ■

The open loop dynamics of the Euler angle-axis representation [2] are given by $\dot{\theta} = \boldsymbol{\lambda}'\bar{\mathcal{R}}(\hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}})$, and $\dot{\boldsymbol{\lambda}} = \frac{1}{2}\left(\mathbf{I} - \frac{\sin(\theta)}{1-\cos(\theta)}[\boldsymbol{\lambda}\times]\right)[\boldsymbol{\lambda}\times]\bar{\mathcal{R}}(\hat{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}})$. The closed loop dynamics are straightforward from (10)

$$\dot{\theta} = -K_\omega \sin(\theta)\boldsymbol{\lambda}'\mathbf{P}\boldsymbol{\lambda}, \tag{13a}$$

$$\dot{\boldsymbol{\lambda}} = K_\omega [\boldsymbol{\lambda}\times][\boldsymbol{\lambda}\times]\mathbf{P}\boldsymbol{\lambda}, \tag{13b}$$

where it is clear that the dynamics of $\boldsymbol{\lambda}$ are autonomous. The closed loop dynamics (13) show that $\mathcal{C}_{V_{\mathcal{R}}}$ is invariant and, from LaSalle's invariance principle, the attitude error converges to the set $\mathcal{C}_{V_{\mathcal{R}}}$. To show that, in fact, the trajectories of the closed loop converge to the origin $\tilde{\mathcal{R}} = \mathbf{I}$ for any initial condition outside a zero measure set, the notion of stability is relaxed by introducing the definitions of region of attraction and almost global stability [1], [6].

Definition 1 (Region of Attraction): Consider the autonomous system $\dot{\mathbf{x}} = f(\mathbf{x})$ evolving on a smooth manifold $\mathcal{M}$, where $\mathbf{x} \in \mathcal{M}$ and $f : \mathcal{M} \to T\mathcal{M}$ is a locally Lipschitz manifold map. Suppose that $\mathbf{x} = \mathbf{x}^*$ is an asymptotically stable equilibrium point of the system. The region of attraction for $\mathbf{x}^*$ is defined as

$$R_A = \{\mathbf{x}_0 \in \mathcal{M} : \phi(t, \mathbf{x}_0) \to \mathbf{x}^* \text{as } t \to \infty\},$$

where $\phi(t, \mathbf{x}_0)$ denotes the solution of the system with initial condition $\mathbf{x} = \mathbf{x}_0$.

For any continuous state feedback law, the region of attraction of a stable equilibrium point is homeomorphic to some Euclidean vector space [7]. A topological limitation to global stability stems from the fact that $\mathrm{SO}(3)$ is not homeomorphic to any Euclidean vector space, which implies that $R_A \neq \mathrm{SO}(3)$ for any equilibrium point candidate.

Definition 2 (Almost GAS / GES [1], [6]): Consider the autonomous system $\dot{\mathbf{x}} = f(\mathbf{x})$ evolving on a smooth manifold $\mathcal{M}$, where $\mathbf{x} \in \mathcal{M}$ and $f : \mathcal{M} \to T\mathcal{M}$ is a locally Lipschitz manifold map. The equilibrium point $\mathbf{x} = \mathbf{x}^*$ is said to be almost globally asymptotically stable if it stable and $\mathcal{M} \setminus R_A$ is a set of zero measure. If $\mathbf{x} = \mathbf{x}^*$ is also exponentially stable, it is almost globally exponentially stable.

It is also necessary to define the distance on $\mathrm{SO}(3)$, inherited by the Euclidean norm [10], $d(\mathcal{R}_1, \mathcal{R}_2) := \frac{1}{2}\|\mathcal{R}_1 - \mathcal{R}_2\|_2$. The distance of $\tilde{\mathcal{R}}$ to the identity matrix $\mathbf{I}$, denoted as $l_I(\tilde{\mathcal{R}}) := d(\tilde{\mathcal{R}}, \mathbf{I})$, is given by $l_I^2(\tilde{\mathcal{R}}) = 2(1 - \cos(\theta))$. Almost global exponential stability of the origin is shown in the following theorem.

Theorem 3: The closed-loop system (12) has an almost GES equilibrium point at $\tilde{\mathcal{R}} = \mathbf{I}$, that is, for any $\tilde{\mathcal{R}}(t_0) \in \mathcal{R}_A$ there exist positive constants $k_{\mathcal{R}}$ and $\lambda_{\mathcal{R}}$ such that the solution of the system (12) satisfies

$$l_I(\tilde{\mathcal{R}}(t)) \leq k_{\mathcal{R}} e^{-\lambda_{\mathcal{R}}(t-t_0)} l_I(\tilde{\mathcal{R}}(t_0)). \tag{14}$$

The region of attraction is given by

$$\begin{aligned} R_A &= \mathrm{SO}(3) \setminus \{\tilde{\mathcal{R}} \in \mathrm{SO}(3) : \mathrm{tr}(\mathbf{I} - \tilde{\mathcal{R}}) = 4\} \\ &= \{(\theta,\boldsymbol{\lambda}) \in \mathrm{D}_\lambda : \theta < \pi\}. \end{aligned}$$

Proof: Define the Lyapunov function

$$W_{\mathcal{R}} := \frac{1-\cos(\theta)}{2}, \quad \dot{W}_{\mathcal{R}} = -\frac{K_\omega}{2}\sin^2(\theta)\boldsymbol{\lambda}'\mathbf{P}\boldsymbol{\lambda}. \quad (15)$$

The set of points where $\dot{W}_{\mathcal{R}} = 0$ is given by

$$C_W = \{\tilde{\mathcal{R}} \in \mathrm{SO}(3) : \tilde{\mathcal{R}} = \mathbf{I} \vee \tilde{\mathcal{R}} = \mathrm{rot}(\pi, \boldsymbol{\lambda})\}.$$

Since $\dot{W}_{\mathcal{R}} \leq 0$, the set contained in a Lyapunov function surface

$$\Omega_\rho = \{\tilde{\mathcal{R}} \in \mathrm{SO}(3) : W_{\mathcal{R}} \leq \rho\}$$

is positively invariant [3], [6], that is, any trajectory starting at t_0 in Ω_ρ satisfies $\tilde{\mathcal{R}}(t) \in \Omega_\rho$ for all $t > t_0$, where the time variable t is explicitly denoted for the sake of clarity.

Note that $W_{\mathcal{R}} < 1 \Rightarrow \theta < \pi$. For any $\rho < 1$, the Lyapunov function is strictly decreasing in Ω_ρ, which implies that $-[1+\cos(\theta(t))] < -[1+\cos(\theta(t_0))] < 0$. Rewriting the Lyapunov function time derivative as $\dot{W}_{\mathcal{R}} = -K_\omega(1+\cos(\theta))\boldsymbol{\lambda}'\mathbf{P}\boldsymbol{\lambda}W_{\mathcal{R}}$, and applying the comparison lemma [6] yields

$$\dot{W}(\tilde{\mathcal{R}}(t)) \leq -K_\omega\left[1+\cos(\theta(t_0))\right]\sigma_1(\mathbf{P})W(\tilde{\mathcal{R}}(t)) \Rightarrow$$
$$W(\tilde{\mathcal{R}}(t)) \leq e^{-2\lambda_{\mathcal{R}}(t-t_0)}W(\tilde{\mathcal{R}}(t_0))$$

where $\lambda_{\mathcal{R}} = \frac{1}{2}K_\omega\left[1+\cos(\theta(t_0))\right]\sigma_1(\mathbf{P})$, and $\sigma_1(\mathbf{P})$ is the smallest singular value of $\mathbf{P}$. Using $W_{\mathcal{R}} = \frac{1}{4}l_I^2(\tilde{\mathcal{R}})$ produces (14) for $\theta(t_0) < \pi$.

Given the closed loop dynamics (13b), it is straightforward to show that $\theta(t_0) = \pi \Rightarrow \dot{\theta} = 0$ so the set $C_W \setminus \{\mathbf{I}\}$, which corresponds to the boundary of R_A, is positively invariant. ■

C. Position Feedback Law

The methodology adopted to derive the attitude feedback law is repeated for the position case. It is immediate that V_p is positive definite and $V_p = 0$ if and only if $\tilde{\mathbf{p}} = 0$. The position feedback law for the system (5) is defined as

$$\begin{aligned}\hat{\mathbf{v}} &= \mathbf{v}_r + ([\boldsymbol{\omega}_r\times] - K_v\mathbf{I})\tilde{\mathbf{p}} - [\hat{\mathbf{p}}\times](\hat{\boldsymbol{\omega}} - \boldsymbol{\omega}_r) \\ &= \bar{\mathbf{v}} + ([\bar{\boldsymbol{\omega}}\times] - K_v\mathbf{I})\mathbf{s}_v + K_\omega[\hat{\mathbf{p}}\times]\mathbf{s}_\omega, \quad (16)\end{aligned}$$

where the feedback term is

$$\mathbf{s}_v := \tilde{\mathbf{p}}, \quad (17)$$

and K_v is a positive scalar. The position feedback law produces a closed loop linear time-invariant system

$$\dot{\tilde{\mathbf{p}}} = -K_v\tilde{\mathbf{p}}, \quad (18)$$

and the exponential stability of the origin is straightforward.

Theorem 4: The equilibrium point $\tilde{\mathbf{p}} = 0$ of the position error dynamics (18) is globally exponentially stable in $\mathbb{R}^3$.

Proof: Exponential convergence to the origin is immediate from the solution of the linear time invariant system (18), given by $\tilde{\mathbf{p}}(t) = e^{-\lambda_p(t-t_0)}\tilde{\mathbf{p}}(t_0)$, where $\lambda_p = K_v$. ■

In some applications, it is necessary to estimate the position with respect to the origin of a specific coordinate frame $\{E\}$, described by ${}^B\hat{\mathbf{p}}_E = \hat{\mathbf{p}} + {}^E_B\hat{\mathbf{R}}'{}^E\mathbf{t}_L$, where ${}^E\mathbf{t}_L$ represents the coordinates of the origin of $\{L\}$ with respect to $\{E\}$, expressed in $\{E\}$. Without loss of generality, the orientations of $\{E\}$ and $\{L\}$ satisfy ${}^E_B\hat{\mathbf{R}} = {}^L_B\hat{\mathbf{R}}$.

As presented in the following proposition, the estimation error ${}^B\tilde{\mathbf{p}}_E = \tilde{\mathbf{p}} + {}^E_B\bar{\mathbf{R}}'(\mathbf{I} - \tilde{\mathcal{R}}'){}^E\mathbf{t}_L$ converges exponentially fast to the origin. The proof is based on well known properties of the Euclidean norm, and omitted due to space constraints.

Proposition 5: The position estimation error ${}^B\tilde{\mathbf{p}}_E$ converges to the origin, and is bounded by an exponentially decreasing term

$$\|{}^B\tilde{\mathbf{p}}_E(t)\| \leq e^{-\lambda_1(t-t_0)}c_1$$

where $\lambda_1 = \min\{\lambda_p, \lambda_{\mathcal{R}}\}$ and $c_1 = \|\tilde{\mathbf{p}}(t_0)\| + l_I(\tilde{\mathcal{R}}(t_0))\|{}^E\mathbf{t}_L\|$.

D. Velocity Bias

In this section, we extend the previous results for the case where the velocity sensor readings are corrupted by a bias term. In this setup, the velocity readings are described by

$$\mathbf{v}_r = \bar{\mathbf{v}} + \bar{\mathbf{b}}_v$$

where the nominal bias is considered constant, $\dot{\bar{\mathbf{b}}}_v = \mathbf{0}$. The proposed Lyapunov function (6) is augmented to account for the effect of the velocity bias

$$V_{b_v} = \frac{n}{2}\tilde{\mathbf{p}}'\tilde{\mathbf{p}} + \frac{1}{2}\tilde{\mathbf{b}}_v'\mathbf{W}_{b_v}\tilde{\mathbf{b}}_v,$$

where $\tilde{\mathbf{b}}_v = \hat{\mathbf{b}}_v - \bar{\mathbf{b}}_v$ is the bias compensation error, $\hat{\mathbf{b}}_v$ is the estimated bias and $\mathbf{W}_{b_v}$ is a positive definite matrix. Clearly, V_{b_v} has an unique global minimum at $(\tilde{\mathbf{p}}, \tilde{\mathbf{b}}_v) = (0, 0)$.

The feedback law for the linear velocity is given by compensating the bias of the velocity reading in (16), producing

$$\begin{aligned}\hat{\mathbf{v}} &= \mathbf{v}_r - \hat{\mathbf{b}}_v + ([\boldsymbol{\omega}_r\times] - K_v\mathbf{I})\tilde{\mathbf{p}} - [\hat{\mathbf{p}}\times](\hat{\boldsymbol{\omega}} - \boldsymbol{\omega}_r) \\ &= \bar{\mathbf{v}} - \tilde{\mathbf{b}}_v - K_v\mathbf{s}_v + [\hat{\mathbf{p}}\times]\mathbf{s}_\omega + [\bar{\boldsymbol{\omega}}\times]\tilde{\mathbf{p}}. \quad (19)\end{aligned}$$

Using the linear velocity feedback law (19), the augmented Lyapunov function dynamics are

$$\dot{V}_{b_v} = -K_v n\tilde{\mathbf{p}}'\tilde{\mathbf{p}} + (\dot{\tilde{\mathbf{b}}}_v'\mathbf{W}_{b_v} - n\tilde{\mathbf{p}}')\tilde{\mathbf{b}}_v.$$

Noting that $\dot{\hat{\mathbf{b}}}_v = \dot{\tilde{\mathbf{b}}}_v$, the bias feedback law is defined as

$$\dot{\hat{\mathbf{b}}}_v = K_{b_v}\tilde{\mathbf{p}} = K_{b_v}\mathbf{s}_v,$$

and $\mathbf{W}_{b_v} = \frac{n}{K_{b_v}}\mathbf{I}$ where K_{b_v} is a positive scalar. The resulting closed loop dynamics are autonomous and given by

$$\dot{\tilde{\mathbf{p}}} = -\tilde{\mathbf{b}}_v - K_v\tilde{\mathbf{p}}, \qquad \dot{\tilde{\mathbf{b}}}_v = K_{b_v}\tilde{\mathbf{p}}. \quad (20)$$

and the Lyapunov function dynamics are described by $\dot{V}_{b_v} = -K_v n\tilde{\mathbf{p}}'\tilde{\mathbf{p}} < 0$.

Theorem 6: The equilibrium point $(\tilde{\mathbf{p}}, \tilde{\mathbf{b}}_v) = (0, 0)$ of the closed-loop position and bias error dynamics (20) is globally exponentially stable.

Proof: The set of points where $\dot{V}_{b_V} = 0$ is given by $\mathcal{C}_{V_{b\,v}} = \{(\tilde{\mathbf{p}}, \tilde{\mathbf{b}}_v) \in \mathbb{R}^3 \times \mathbb{R}^3 : \tilde{\mathbf{p}} = 0\}$. To show asymptotic stability, we apply LaSalle's invariance principle [6]. The closed loop system (20) satisfies $\tilde{\mathbf{p}} \in \mathcal{C}_{V_{b\,v}} \Rightarrow \dot{\tilde{\mathbf{p}}} = 0 \Rightarrow \tilde{\mathbf{b}}_v = 0$, so the largest invariant set in $\mathcal{C}_{V_{b\,v}}$ is $\{(0, 0)\}$. Exponential stability of the origin is a direct consequence of the solution of linear time-invariant systems [6]. ■

IV. OBSERVER PROPERTIES

This section points out important characteristics of the observer. Namely, it is shown that the position and attitude feedback laws can be expressed as an explicit function of the sensor readings. To gain further insight on how the solutions of the attitude system evolve, we analyze the closed-loop trajectories of the attitude error dynamics in the Euler angle-axis representation.

A. Output Feedback

The feedback terms formulated in (11) and (17) are functions of the nominal attitude $\bar{\mathcal{R}}$ and position $\bar{\mathbf{p}}$, which are not available directly from the landmark readings. In this section, it is shown that the position and attitude feedback laws can be expressed as an explicit function of the landmark readings.

Theorem 7: The feedback laws (10) and (19) are explicit functions of the sensor readings and state estimates

$$\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}_r - K_\omega \mathbf{s}_\omega,$$
$$\hat{\mathbf{v}} = \mathbf{v}_r - \hat{\mathbf{b}}_v + ([\boldsymbol{\omega}_r \times] - K_v \mathbf{I})\mathbf{s}_v + K_\omega [\hat{\mathbf{p}}\times]\mathbf{s}_\omega,$$

where $\mathbf{s}_\omega = \sum_{i=1}^n (\hat{\mathbf{x}}_i \times \mathbf{q}_{r\,i})$ and $\mathbf{s}_v = \hat{\mathbf{p}} + \frac{1}{n}\sum_{i=1}^n \mathbf{q}_{r\,i}$.

Proof: Using the centroid invariance with respect to coordinate frame orientation (3), it is straightforward to show that $\mathbf{s}_v = \hat{\mathbf{p}} + \frac{1}{n}\sum_{i=1}^n \bar{\mathbf{q}}_i$.

Using $\mathcal{R}[\mathbf{K}\otimes] = [\mathcal{R}\mathbf{K}\mathcal{R}'\otimes]$, where $\mathbf{K} = -\mathbf{K}'$, and the rotation error definition yields $\mathbf{s}_\omega = \left[{}^B\bar{\mathbf{X}}{}^B\hat{\mathbf{X}}' - {}^B\hat{\mathbf{X}}{}^B\bar{\mathbf{X}}'\otimes\right]$, where ${}^B\hat{\mathbf{X}} := \hat{\mathcal{R}}'\mathbf{X}$ and ${}^B\bar{\mathbf{X}} := \bar{\mathcal{R}}'\mathbf{X}$. Using $\sum_{i=1}^n \hat{\mathbf{x}}_i\bar{\mathbf{q}}_i' = {}^B\hat{\mathbf{X}}{}^B\bar{\mathbf{X}}'$ and $\bar{\mathbf{q}}_i\hat{\mathbf{x}}_i' - \hat{\mathbf{x}}_i\bar{\mathbf{q}}_i' = [(\hat{\mathbf{x}}_i \times \bar{\mathbf{q}}_i)\times]$ yields $\mathbf{s}_\omega = \sum_{i=1}^n (\hat{\mathbf{x}}_i \times \bar{\mathbf{q}}_i)$. ■

B. Euler Angle-Axis Parametrization Dynamics

The closed loop trajectories of the system in the Euler angle-axis parametrization are analyzed.

Theorem 8: The origin of the system (13a) is asymptotically stable, with region of attraction described by $\{\theta \in [0,\pi] : \theta < \pi\}$, and θ decreases monotonically.

Assuming that the singular values of $\mathbf{P}$ satisfy $\sigma_1(\mathbf{P}) < \sigma_2(\mathbf{P}) < \sigma_3(\mathbf{P})$, the asymptotic convergence of the Euler axis is described by

$$\begin{cases} \boldsymbol{\lambda}(t) \to \operatorname{sgn}(\mathbf{u}_1'\boldsymbol{\lambda}(t_0))\mathbf{u}_1 \text{ as } t \to \infty, & \text{if } \mathbf{u}_1'\boldsymbol{\lambda}(t_0) \neq 0 \\ \boldsymbol{\lambda}(t) \to \{\mathbf{u}_2, \mathbf{u}_3\} \text{ as } t \to \infty, & \text{if } \mathbf{u}_1'\boldsymbol{\lambda}(t_0) = 0 \end{cases}.$$

where $\mathbf{u}_i$ is the unitary eigenvector associated with $\sigma_i(\mathbf{P})$.

Proof: The region of attraction of $\theta = 0$ is immediate from Proposition 3. The Lyapunov function (15) is strictly decreasing in R_A, and $\forall_{t_2,t_1}\ W(\tilde{\mathcal{R}}(t_2)) < W(\tilde{\mathcal{R}}(t_1)) \Rightarrow \theta(t_2) < \theta(t_1)$, so $\theta(t)$ converges monotonically to the origin.

The rotation vector dynamics (13b) are autonomous. Define the Lyapunov function

$$V_s = 1 + s\mathbf{u}_1'\boldsymbol{\lambda}, \qquad \dot{V}_s = s\mathbf{u}_1'\boldsymbol{\lambda}\left(\boldsymbol{\lambda}\mathbf{P}\boldsymbol{\lambda} - \sigma_1(\mathbf{P})\right),$$

in the domain $\mathrm{S}(2)$, where $s \in \{-1, +1\}$. From the Schwartz inequality, the Lyapunov function is positive definite and $V_s = 0 \Leftrightarrow \boldsymbol{\lambda} = -s\mathbf{u}_1$. Assuming that the eigenvalue has multiplicity 1, the set of point where $\dot{V}_s = 0$ is given by

$$C_{V_s} = \{\boldsymbol{\lambda} \in \mathrm{S}(2) : \boldsymbol{\lambda} = \pm\mathbf{u}_1 \vee \mathbf{u}_1'\boldsymbol{\lambda} = 0\}.$$

The Lyapunov time derivative $\dot{V}_{+1}$ and $\dot{V}_{-1}$ is indefinite in the domain $\mathrm{S}(2)$. For the initial conditions $s\mathbf{u}_1'\boldsymbol{\lambda}(t_0) < 0$ define $0 < \beta < 1$ such that $s\mathbf{u}_1'\boldsymbol{\lambda}(t_0) \leq \beta - 1$. The level sets $\Omega_\beta^s = \{\boldsymbol{\lambda} \in \mathrm{S}(2) : V_s(\boldsymbol{\lambda}) \leq \beta\}$ are positively invariant. The unique points where $\dot{V}_s = 0$ in Ω_β^s, given by $\boldsymbol{\lambda} = -s\mathbf{u}_1$, are asymptotically stable.

To analyze the case $\mathbf{u}_1'\boldsymbol{\lambda}(t_0) = 0$, the property $\mathbf{u}_1'\boldsymbol{\lambda} = 0 \Rightarrow \mathbf{u}_1'\dot{\boldsymbol{\lambda}} = 0$ shows that the set defined by $\mathbf{u}_1'\boldsymbol{\lambda} = 0$ is positively invariant. The trajectories of $\boldsymbol{\lambda}$ (and hence the positive limit set) are independent of θ, so Lemma 2 implies that $\boldsymbol{\lambda}(t) \to \{\mathbf{u}_2, \mathbf{u}_3\}$ as $t \to \infty$. ■

The asymptotic convergence for the specific case $\exists_{i\neq j}\sigma_i(\mathbf{P}) = \sigma_j(\mathbf{P})$ can be obtained by following the same steps of the proof of Theorem 8. In particular, if $\exists_\sigma \mathbf{P} = \sigma\mathbf{I}$, then the solution to (13b) is given by $\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}(t_0)$.

V. SIMULATIONS

In this section, the proposed attitude and position observer properties are illustrated in simulation. The landmarks are placed on the xy plane ${}^L\bar{\mathbf{x}}_1 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}'$, ${}^L\bar{\mathbf{x}}_2 = \begin{bmatrix}\frac{1}{2} & -\frac{1}{2} & 0\end{bmatrix}'$ and ${}^L\bar{\mathbf{x}}_3 = \begin{bmatrix}-\frac{1}{2} & -\frac{1}{2} & 0\end{bmatrix}'$, which satisfies the conditions expressed in Assumption 1. The singular values of $\mathbf{P}$ are all distinct. The smallest one is characterized by $\sigma_1(\mathbf{P}) = 0.5$ and is associated with the eigenvector $\mathbf{u}_1 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}'$. The feedback gains are given by $K_p = K_\omega = 1$, and the rigid body trajectory is computed using oscillatory angular and linear rates of 1 Hz.

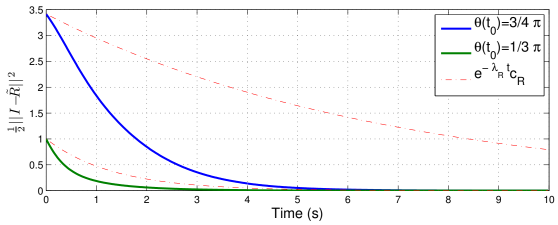
As shown in Fig. 2(a), the attitude error converges exponentially to $\tilde{\mathcal{R}} = \mathbf{I}$. The attitude error trajectory is below the exponential bound (14), that is more conservative for larger values of $\theta(t_0)$.

The position estimation error $\tilde{\mathbf{p}}$ decays exponentially, as illustrated in Fig. 2(b), and the position estimation error with respect to Earth frame ${}^B\tilde{\mathbf{p}}_E$ is bounded by an exponentially decaying term. Using the bias feedback gain $K_{b_v} = 0.5$, the bias compensation error converges to the origin, as depicted in Fig. 2(c).

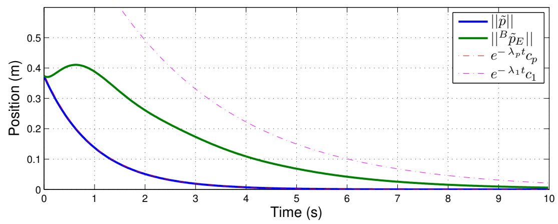
The Euler axis trajectories in the hemisphere $\mathbf{u}_1'\boldsymbol{\lambda} \geq 0$, depicted in Fig. 3, illustrate the asymptotic convergence discussed in Section IV-B. Clearly, $\boldsymbol{\lambda}(t) \to \mathbf{u}_1$ as $t \to \infty$ for $\mathbf{u}_1'\boldsymbol{\lambda} > 0$ and the border $\mathbf{u}_1'\boldsymbol{\lambda} = 0$ is an invariant set.

VI. CONCLUSIONS

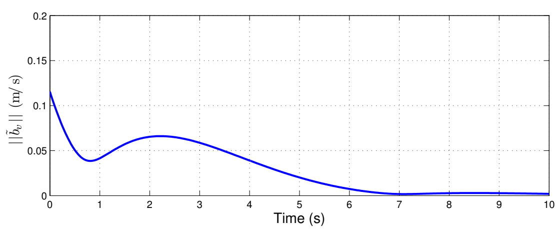
A nonlinear observer for position and attitude estimation on $\mathrm{SE}(3)$ was proposed and almost global exponential stability (GES) of the origin was obtained. The results were extended for the case of biased linear velocity readings. The proposed Lyapunov function defined by the landmark measurement error allowed for i) deriving independent position and attitude feedback laws ii) gaining a geometric insight on the required landmark configuration for attitude determination. Future work will focus on the exact discrete time implementation of the algorithm.



(a) Attitude Estimation Error and Bounds ($\boldsymbol{\lambda}(t_0) = \frac{1}{\sqrt{3}}\,[1\,1\,1]'$)



(b) Position Estimation Errors $\tilde{\mathbf{p}}$ and ${}^B\tilde{\mathbf{p}}_E$, and Bounds



(c) Bias Estimation Error

Fig. 2. Attitude, Position and Bias Estimation Error

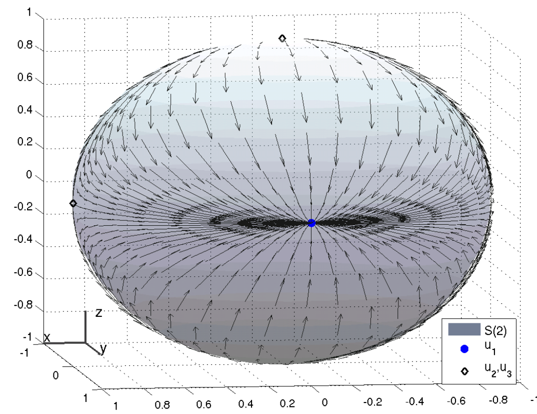


Fig. 3. Euler Axis Trajectories on $\mathrm{S}(2)$

Appendix

A. Definitions and Proofs

In this section, the sets referenced in this work are defined, and the properties of $V_{\mathcal{R}}$ are derived. The set of $n \times m$ matrices with real entries and the subset of square matrices are denoted by $\mathrm{M}(n,m)$ and $\mathrm{M}(n) := \mathrm{M}(n,n)$, respectively. The sets of orthogonal, special orthogonal and special Euclidean matrices are respectively defined as $\mathrm{O}(n) = \{\mathbf{U} \in \mathrm{M}(n) : \mathbf{U}'\mathbf{U} = \mathbf{I}\}$, $\mathrm{SO}(n) = \{\mathbf{R} \in \mathrm{O}(n) : \det(\mathbf{R}) = 1\}$, $\mathrm{SE}(n)$ is the product space [11, p.35] of $\mathrm{SO}(n)$ with $\mathbb{R}^n$, $\mathrm{SE}(n) = \mathrm{SO}(n) \times \mathbb{R}^n$, and the n-dimensional sphere is given by $\mathrm{S}(n) = \{\mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x}'\mathbf{x} = 1\}$.

Proof: [Lemma 1] From (7), the zeros of $V_{\mathcal{R}}$ are $\theta = 0$ or $\boldsymbol{\lambda} \in \mathcal{N}(\mathbf{P})$. The singular value decomposition of $\mathbf{X}$ yields $\mathbf{P} = \mathbf{U}\,\mathrm{diag}\,(s_{22}^2 + s_{33}^2, s_{11}^2 + s_{33}^2, s_{11}^2 + s_{22}^2)\mathbf{U}'$, where $\mathbf{U} \in \mathrm{O}(3)$ and $s_{11} > s_{22} > s_{33}$ are the singular values of $\mathbf{X}$. Then $\mathbf{P} > 0$ if and only if $\{s_{11}, s_{22}\} \neq 0$, i.e. $\mathrm{rank}(\mathbf{X}) \geq 2$. ∎

Proof: [Lemma 2] The points where $\dot{V}_{\mathcal{R}} = 0$ are given by $\mathbf{s}_\omega = 0 \Leftrightarrow \mathbf{Q}'(\theta, \boldsymbol{\lambda})\mathbf{P}\boldsymbol{\lambda} = 0 \Leftrightarrow \sin(\theta)\mathbf{P}\boldsymbol{\lambda} - (1 - \cos(\theta))\,[\boldsymbol{\lambda}\times]\,\mathbf{P}\boldsymbol{\lambda} = 0$. For any $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{x}$ and $[\boldsymbol{\lambda}\times]\,\mathbf{x}$ are noncolinear, so $\mathbf{s}_\omega = 0$ if and only if $\theta \in \{0, \pi\}$. For the $\theta = \pi$ case, $[\boldsymbol{\lambda}\times]\,\mathbf{P}\boldsymbol{\lambda} = 0 \Leftrightarrow \exists_\alpha \mathbf{P}\boldsymbol{\lambda} = \alpha\boldsymbol{\lambda}$, so $\dot{V}_{\mathcal{R}} = 0$ if and only if $\theta = 0 \vee (\theta = \pi \wedge \boldsymbol{\lambda} \in \mathrm{eigvec}(\mathbf{P}))$. ∎

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