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GES Long Baseline Navigation with Unknown Sound Velocity and Discrete-time Range Measurements

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Abstract— **A common assumption in long baseline (LBL) underwater acoustic navigation is that the speed of sound is available. This quantity depends on the medium and it is usually measured or profiled prior to the experiments. This paper proposes a novel filtering solution that explicitly takes into account the estimation of the speed of propagation of the acoustic waves in the medium. Based on discrete-time range measurements, an augmented system is derived that can be regarded as linear for observability and observer design purposes. Its observability is discussed and a Kalman filter provides the estimation solution, with globally exponentially stable (GES) error dynamics. Simulation results are presented, considering noisy measurements, to evaluate the proposed solution, which evidence both fast convergence and good performance.**

I. Introduction

Long baseline (LBL) navigation is a common solution for positioning of underwater vehicles, resorting in general to the round-trip travel time of acoustic signals from the vehicle to several transponders, fixed in known positions in the mission scenario, see e.g. [1], [2], [3], [4], [5], and [6]. In [7] the author proposes a GPS-like system consisting of buoys equipped with Differential GPS. A related solution, denominated as GPS Intelligent Buoy (GIB) system, is now commercially available, see [8]. Further work on the GIB underwater positioning system can be found in [9]. For interesting discussions and detailed surveys on underwater vehicle navigation techniques and challenges see [10], [11], and [12].

In previous work by the authors a novel filtering solution was proposed for long baseline navigation [13], based on an extension of the framework for single range measurements, proposed in [14], to multiple range measurements. A common assumption, present in all previously mentioned contributions, is that the speed of propagation of the waves in the medium is known or measured. This quantity depends on several characteristics such as the salinity, pressure, and temperature and it is either measured or profiled, often prior to the experiments. The main contribution of this paper is the development of a novel framework for long baseline navigation that explicitly includes the estimation of the speed of propagation of the acoustic waves in the medium. Based on discrete-time range measurements, combined with attitude and relative velocity readings obtained at high rates, an augmented system is derived that can be considered as linear for observability and observer design purposes. Its observability is carefully analyzed and a Kalman filter is considered as the estimation solution, with globally exponentially stable error dynamics.

The paper is organized as follows. The problem statement and the nominal system dynamics are introduced in Section II, while the filter design is detailed in Section III. Simulation results are presented in Section IV and Section V summarizes the main results of the paper.

A. Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. A block diagonal matrix is represented by $\operatorname{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$. For $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \cdot \mathbf{y}$ represents the inner product.

II. Problem statement

Consider a standard Long Baseline acoustic positioning system, consisting of a set of transponders that are fixed in the mission scenario, where an underwater vehicle operates, also equipped with an acoustic transponder.Typically, the transponder of the vehicle sends a known acoustic signal to interrogate the transponders of the Long Baseline acoustic positioning system, which then respond sending each a known acoustic signal. These signals are then received by the transponder of the vehicle and the range is usually calculated using the round-trip travel time and the speed of propagation of the acoustic waves in the medium. In this paper, the latter is assumed unknown and as such the range measurements, which are measured periodically, are only available up to a scaling factor. Further suppose that the vehicle is equipped with an Attitude and Heading Reference System (AHRS) and a Doppler Velocity Log (DVL). The problem considered herein is that of designing a continuous-discrete filter, with globally exponentially stable error dynamics, to estimate the position and linear velocity of the vehicle, as well as the speed of propagation of the acoustic waves in the medium.

A. System dynamics

Let $\{I\}$ denote a local inertial reference coordinate frame and $\{B\}$ a coordinate frame attached to the vehicle, usually referred to as the body-fixed reference frame. The linear motion of the vehicle satisfies

$$\dot{\mathbf{p}}(t) = \mathbf{R}(t)\mathbf{v}(t), \tag{1}$$

where $\mathbf{p}(t) \in \mathbb{R}^3$ denotes the inertial position of the vehicle, $\mathbf{v}(t) \in \mathbb{R}^3$ is the velocity of the vehicle relative to $\{I\}$, expressed in body-fixed coordinates, and $\mathbf{R}(t) \in SO(3)$ is the rotation matrix from $\{B\}$ to $\{I\}$.

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The AHRS provides the rotation matrix $\mathbf{R}(t)$, while the DVL measures, in the absence of bottom-lock, the velocity of the vehicle relative to the fluid, expressed in body-fixed coordinates. Let $\mathbf{v}_c(t) \in \mathbb{R}^3$ denote the velocity of the fluid, in inertial coordinates, and $\mathbf{v}_r(t) \in \mathbb{R}^3$ be the DVL reading, i.e., the velocity of the vehicle relative to the fluid, expressed in body-fixed coordinates. Then,

$$\mathbf{v}(t) = \mathbf{v}_r(t) + \mathbf{R}^T(t)\mathbf{v}_c(t). \tag{2}$$

Finally, let $\mathbf{s}_i \in \mathbb{R}^3$, $i = 1, \ldots, L$, denote the inertial positions of the transponders. Then, the range measurements are given by

$$r_i(k) = v_s(t_k) \|\mathbf{s}_i - \mathbf{p}(t_k)\|, \tag{3}$$

with $t_k := t_0 + kT$, $k \in \mathbb{N}$, where $T > 0$ is the sampling period, t_0 is the initial time, and $v_s(t) > 0$ is a scaling factor that accounts for the unknown speed of propagation of the acoustic waves in the medium.

Assuming that both the fluid velocity and the speed of propagation of the acoustic waves in the medium are constant, i.e., $\dot{\mathbf{v}}_c(t) = \mathbf{0}$ and $\dot{v}_s(t) = 0$, and combining (1)-(3), results in the nonlinear system with discrete outputs

$$\begin{cases} \dot{\mathbf{p}}(t) = \mathbf{v}_c(t) + \mathbf{R}(t)\mathbf{v}_r(t) \\ \dot{\mathbf{v}}_c(t) = \mathbf{0} \\ \dot{v}_s(t) = 0 \\ r_1(k) = v_s(t_k) \|\mathbf{s}_1 - \mathbf{p}(t_k)\| \\ \vdots \\ r_L(k) = v_s(t_k) \|\mathbf{s}_L - \mathbf{p}(t_k)\| \end{cases}. \tag{4}$$

The problem considered herein is the design of an estimator for (4) with globally exponentially stable error dynamics.

III. Filter Design

In previous work by the authors, [13], a novel LBL framework was proposed, in continuous time. In short, additional states and outputs are derived that allow to consider the system as linear in the state, even though it still is, in fact, nonlinear. This is done by means of identification of some nonlinear terms as new variables and noticing that the output and input are available signals for observer design purposes. In this paper, a similar approach is somehow pursued but considering: i) discrete-time measurements; and ii) scaled ranges, with unknown speed of propagation of the acoustic waves in the medium. This setting leads to a different state vector and consequently a different dynamic system, and captures the nature of the underwater ranging sensing system when the speed of propagation is unknown or only known approximately.

A. Discretization and system augmentation

The exact discrete-time system dynamics corresponding to (4) are given by

$$\begin{cases} \mathbf{p}(t_{k+1}) = \mathbf{p}(t_k) + T\mathbf{v}_c(t_k) + \int_{t_k}^{t_{k+1}} \mathbf{R}(\tau)\mathbf{v}_r(\tau)\, d\tau \\ \mathbf{v}_c(t_{k+1}) = \mathbf{v}_c(t_k) \\ v_s(t_{k+1}) = v_s(t_k) \\ r_1(k) = v_s(t_k) \|\mathbf{s}_1 - \mathbf{p}(t_k)\| \\ \vdots \\ r_L(k) = v_s(t_k) \|\mathbf{s}_L - \mathbf{p}(t_k)\| \end{cases}. \tag{5}$$

Define the discrete-time states

$$\begin{cases} \mathbf{x}_1(k) := v_s^2(t_k)\, \mathbf{p}(t_k) \\ \mathbf{x}_2(k) := v_s^2(t_k)\, \mathbf{v}_c(t_k) \\ x_3(k) = v_s^2(t_k) \end{cases}.$$

From (5) one may write

$$\begin{cases} \mathbf{x}_1(k+1) = \mathbf{x}_1(k) + T\mathbf{x}_2(k) + x_3(k)\,\mathbf{u}(k) \\ \mathbf{x}_2(k+1) = \mathbf{x}_2(k) \\ x_3(k+1) = x_3(k) \end{cases}, \tag{6}$$

where

$$\mathbf{u}(k) := \int_{t_k}^{t_{k+1}} \mathbf{R}(\tau)\, \mathbf{v}_r(\tau)\, d\tau.$$

Now, consider the scaled range measurements as additional system states, i.e., define

$$\begin{cases} x_4(k) := r_1(k) \\ \vdots \\ x_{3+L}(k) := r_L(k) \end{cases}.$$

To derive the discrete-time dynamics of the range measurements, consider their squares and expand

$$r_i^2(k+1) = x_3(k+1) \left\| \mathbf{s}_i - \frac{\mathbf{x}_1(k+1)}{x_3(k+1)} \right\|^2$$

using (6), which gives

$$\begin{aligned} r_i^2(k+1) \;=\; & 2\mathbf{u}(k) \cdot \mathbf{x}_1(k) - 2T\left[\mathbf{s}_i - \mathbf{u}(k)\right] \cdot \mathbf{x}_2(k) \\ & - \left[2\mathbf{s}_i - \mathbf{u}(k)\right] \cdot \mathbf{u}(k)\, x_3(k) \\ & + r_i^2(k) + 2T\frac{\mathbf{x}_1(k) \cdot \mathbf{x}_2(k)}{x_3(k)} \\ & + T^2 \frac{\|\mathbf{x}_2(k)\|^2}{x_3(k)}, \end{aligned} \tag{7}$$

$i = 1, \ldots, L$. Identifying the nonlinear terms $\mathbf{x}_1(k) \cdot \mathbf{x}_2(k) / x_3(k)$ and $\|\mathbf{x}_2(k)\|^2 / x_3(k)$ in (7) with new system states, i.e.,

$$\begin{cases} x_{4+L}(k) := \frac{\mathbf{x}_1(k) \cdot \mathbf{x}_2(k)}{x_3(k)} = v_s^2(t_k)\, \mathbf{p}(t_k) \cdot \mathbf{v}_c(t_k) \\ x_{5+L}(k) := \frac{\|\mathbf{x}_2(k)\|^2}{x_3(k)} = v_s^2(t_k) \|\mathbf{v}_c(t_k)\|^2 \end{cases}, \tag{8}$$

and noticing that $r_i^2(k) = x_{i+3}(k) r_i(k)$, $i = 1, \ldots, L$, allows to write

$$\begin{aligned} x_{i+3}(k+1) \;=\; & \frac{2\mathbf{u}(k) \cdot \mathbf{x}_1(k)}{r_i(k+1)} - \frac{2T\left[\mathbf{s}_i - \mathbf{u}(k)\right] \cdot \mathbf{x}_2(k)}{r_i(k+1)} \\ & - \frac{\left[2\mathbf{s}_i - \mathbf{u}(k)\right] \cdot \mathbf{u}(k)}{r_i(k+1)} x_3(k) \\ & + \frac{r_i(k)\, x_{i+3}(k)}{r_i(k+1)} + \frac{2T x_{4+L}(k)}{r_i(k+1)} \\ & + \frac{T^2 x_{5+L}(k)}{r_i(k+1)}, \end{aligned}$$

$i = 1, \ldots, L$. The evolution of the new states can be written, using (6), as

$$\begin{cases} x_{4+L}(k+1) = \mathbf{u}(k) \cdot \mathbf{x}_2(k) + x_{4+L}(k) + T x_{5+L}(k) \\ x_{5+L}(k+1) = x_{5+L}(k) \end{cases}.$$

Define the augmented state vector as

$$\mathbf{x}(k) := \begin{bmatrix} \mathbf{x}_1(k) \\ \mathbf{x}_2(k) \\ x_3(k) \\ x_4(k) \\ \vdots \\ x_{3+L}(k) \\ x_{4+L}(k) \\ x_{5+L}(k) \end{bmatrix} \in \mathbb{R}^{3+3+1+L+2}.$$

Then, the discrete-time system dynamics can be written as

$$\mathbf{x}(k+1) = \mathbf{A}(k)\,\mathbf{x}(k),$$

where $\mathbf{A}(k) \in \mathbb{R}^{(7+L+2)\times(7+L+2)}$,

$$\mathbf{A}(k) = \left[\begin{array}{ccc|c|cc} \mathbf{I} & T\mathbf{I} & \mathbf{u}(k) & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & 0 & 0 \\ \hline & & & & \frac{2T}{r_1(k+1)} & \frac{T^2}{r_1(k+1)} \\ & \mathbf{A}_{21}(k) & & \mathbf{A}_{22}(k) & \vdots & \vdots \\ & & & & \frac{2T}{r_L(k+1)} & \frac{T^2}{r_L(k+1)} \\ \hline \mathbf{0} & \mathbf{u}^T(t) & 0 & \mathbf{0} & 1 & T \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & 0 & 1 \end{array}\right],$$

with $\mathbf{A}_{21}(k) \in \mathbb{R}^{L\times 7}$

$$\mathbf{A}_{21}(k) = \begin{bmatrix} \frac{2\mathbf{u}^T(k)}{r_1(k+1)} & 2T\frac{\mathbf{u}^T(k)-\mathbf{s}_1^T}{r_1(k+1)} & \frac{\mathbf{u}(k)-2\mathbf{s}_1}{r_1(k+1)}\cdot\mathbf{u}(k) \\ \vdots & \vdots & \vdots \\ \frac{2\mathbf{u}^T(k)}{r_L(k+1)} & 2T\frac{\mathbf{u}^T(k)-\mathbf{s}_L^T}{r_L(k+1)} & \frac{\mathbf{u}(k)-2\mathbf{s}_L}{r_L(k+1)}\cdot\mathbf{u}(k) \end{bmatrix},$$

and

$$\mathbf{A}_{22}(k) = \mathbf{diag}\left(\frac{r_1(k)}{r_1(k+1)},\ldots,\frac{r_L(k)}{r_L(k+1)}\right) \in \mathbb{R}^{L\times L}.$$

To grasp the LBL structure, take the difference of the squares of range measurements to two different transponders, which gives

$$r_i^2(k) - r_j^2(k) = v_s^2(t_k)\left(\|\mathbf{s}_i\|^2 - \|\mathbf{s}_j\|^2\right) - 2v_s^2(t_k)\left[(\mathbf{s}_i - \mathbf{s}_j)\cdot\mathbf{p}(t_k)\right]. \quad (9)$$

Using

$$r_i^2(k) - r_j^2(k) = \left[r_i(k) + r_j(k)\right]\left[x_{3+i}(k) - x_{3+j}(k)\right],$$

together with (8), allows to rewrite (9) as

$$\frac{2(\mathbf{s}_i-\mathbf{s}_j)}{r_i(k)+r_j(k)}\cdot\mathbf{x}_1(k) - \frac{\|\mathbf{s}_i\|^2-\|\mathbf{s}_j\|^2}{r_i(k)+r_j(k)}x_3(k) + x_{3+i}(k) - x_{3+j}(k) = 0 \quad (10)$$

for $i, j \in \{1,\ldots,L\}$, $i \neq j$. Discarding the original nonlinear output equation, considering that the states $x_4(k),\ldots,x_{3+L}(k)$ are measured, and using (10) allows to define the augmented system output

$$\begin{cases} y_1(k) = x_4(k) \\ \vdots \\ y_L(k) = x_{3+L}(k) \\ y_{L+1}(k) = \frac{2(\mathbf{s}_1-\mathbf{s}_2)\cdot\mathbf{x}_1(k)}{r_1(k)+r_2(k)} - \frac{\|\mathbf{s}_1\|^2-\|\mathbf{s}_2\|^2}{r_1(k)+r_2(k)}x_3(k) \\ \qquad +x_{3+1}(k) - x_{3+2}(k) \\ y_{L+2}(k) = \frac{2(\mathbf{s}_1-\mathbf{s}_3)\cdot\mathbf{x}_1(k)}{r_1(k)+r_3(k)} - \frac{\|\mathbf{s}_1\|^2-\|\mathbf{s}_3\|^2}{r_1(k)+r_3(k)}x_3(k) \\ \qquad +x_{3+1}(k) - x_{3+3}(k) \\ \vdots \\ y_{L+C_2^L-1}(k) = \frac{2(\mathbf{s}_{L-2}-\mathbf{s}_L)\cdot\mathbf{x}_1(k)}{r_{L-2}(k)+r_L(k)} - \frac{\|\mathbf{s}_{L-2}\|^2-\|\mathbf{s}_L\|^2}{r_{L-2}(k)+r_L(k)}x_3(k) \\ \qquad +x_{3+L-2} - x_{3+L}(k) \\ y_{L+C_2^L}(k) = \frac{2(\mathbf{s}_{L-1}-\mathbf{s}_L)\cdot\mathbf{x}_1(k)}{r_{L-1}(k)+r_L(k)} - \frac{\|\mathbf{s}_{L-1}\|^2-\|\mathbf{s}_L\|^2}{r_{L-1}(k)+r_L(k)}x_3(k) \\ \qquad +x_{3+L-1} - x_{3+L}(k) \end{cases},$$

where C_2^L is the number of 2-combinations of L elements, i.e. $C_2^L = L(L-1)/2$.

The discrete-time augmented system can then be written, in compact form, as

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}(k)\,\mathbf{x}(k) \\ \mathbf{y}(k+1) = \mathbf{C}(k+1)\,\mathbf{x}(k+1) \end{cases}, \quad (11)$$

with

$$\mathbf{C}(k) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{C}_{21}(k) & \mathbf{C}_{22} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(L+C_2^L)\times(7+L+2)},$$

where $\mathbf{C}_{21}(k) \in \mathbb{R}^{C_2^L\times 7}$ is given by

$$\mathbf{C}_{21}(k) = \begin{bmatrix} \frac{2(\mathbf{s}_1-\mathbf{s}_2)^T}{r_1(k)+r_2(k)} & \mathbf{0} & -\frac{\|\mathbf{s}_1\|^2-\|\mathbf{s}_2\|^2}{r_1(k)+r_2(k)} \\ \frac{2(\mathbf{s}_1-\mathbf{s}_3)^T}{r_1(k)+r_3(k)} & \mathbf{0} & -\frac{\|\mathbf{s}_1\|^2-\|\mathbf{s}_3\|^2}{r_1(k)+r_3(k)} \\ \vdots & \vdots & \vdots \\ \frac{2\left(\mathbf{s}_{L-2}-\mathbf{s}_L\right)^T}{r_{L-2}(k)+r_L(k)} & \mathbf{0} & -\frac{\|\mathbf{s}_{L-2}\|^2-\|\mathbf{s}_L\|^2}{r_{L-2}(k)+r_L(k)} \\ \frac{2\left(\mathbf{s}_{L-1}-\mathbf{s}_L\right)^T}{r_{L-1}(k)+r_L(k)} & \mathbf{0} & -\frac{\|\mathbf{s}_{L-1}\|^2-\|\mathbf{s}_L\|^2}{r_{L-1}(k)+r_L(k)} \end{bmatrix},$$

and

$$\mathbf{C}_{22} = \begin{bmatrix} 1 & -1 & 0 & \ldots & \ldots & \ldots & 0 \\ 1 & 0 & -1 & 0 & \ldots & \ldots & 0 \\ & & & \vdots & & & \\ 0 & \ldots & \ldots & 0 & 1 & 0 & -1 \\ 0 & \ldots & \ldots & \ldots & 0 & 1 & -1 \end{bmatrix} \in \mathbb{R}^{C_2^L\times L}.$$

Remark 1: Notice that the system (11) is well defined as no range measurement can be nonpositive. Indeed, by definition, the range measurements are nonnegative and a null measurement would imply that two transponders were in the same position, which is impossible. In fact, there is always a minimum distance between transponders.

B. Observability analysis

The system (11) can be regarded as a discrete linear time-varying system for observer design purposes, even though the system matrices $\mathbf{A}(k)$ and $\mathbf{C}(k)$ depend on the system input and the range measurements. This is possible because for observer (or filter) design purposes both the ranges and the input are available and, hence, they can be simply considered as functions of time. This idea was first pursued by the authors in [14, Lemma 1] for continuous systems, whose application if equivalent for the discrete-time case, as shown in the following lemma.

Lemma 1: Consider the nonlinear discrete-time system

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{\mathcal{A}}\left(k,\boldsymbol{\mathcal{U}}_{k_0}^{k+1},\boldsymbol{\mathcal{Y}}_{k_0}^{k+1}\right)\mathbf{x}(k) \\ \boldsymbol{y}(k+1) = \boldsymbol{\mathcal{C}}\left(k+1,\boldsymbol{\mathcal{U}}_{k_0}^{k+1},\boldsymbol{\mathcal{Y}}_{k_0}^{k+1}\right)\boldsymbol{x}(k+1) \end{cases}, \quad (12)$$

where $\boldsymbol{\mathcal{U}}_{k_0}^{k_f} := \{\boldsymbol{u}(k_0),\boldsymbol{u}(k_0+1),\ldots,\boldsymbol{u}(k_f-1)\}$ and $\boldsymbol{\mathcal{Y}}_{k_0}^{k_f} := \{\boldsymbol{y}(k_0),\boldsymbol{y}(k_0+1),\ldots,\boldsymbol{y}(k_f-1)\}$ are the input and output signals, respectively, on the time interval $[k_0, k_f[$, and $\boldsymbol{x}(k) \in \mathbb{R}^n$. If $\operatorname{rank}(\boldsymbol{\mathcal{O}}(k_0,k_f)) = n$, where $\boldsymbol{\mathcal{O}}(k_0,k_f)$ is the observability matrix associated with the pair $\left(\boldsymbol{\mathcal{A}}\left(k,\boldsymbol{\mathcal{U}}_{k_0}^{k_f},\boldsymbol{\mathcal{Y}}_{k_0}^{k_f}\right),\boldsymbol{\mathcal{C}}\left(k,\boldsymbol{\mathcal{U}}_{k_0}^{k_f},\boldsymbol{\mathcal{Y}}_{k_0}^{k_f}\right)\right)$ on $\mathcal{I} := [k_0, k_f]$, then the nonlinear system (12) is observable on $\mathcal{I}$ in the sense that, given the system input and output signals $\boldsymbol{\mathcal{U}}_{k_0}^{k_f}$ and $\boldsymbol{\mathcal{Y}}_{k_0}^{k_f}$, the initial condition $\boldsymbol{x}(k_0)$ is uniquely defined.

Proof: The proof follows as in classic linear systems theory, noting that as all signals are available, it is possible to compute the transition matrix and the observability matrix, even though these depend on the system input and output. It is omitted due to space limitations. ■

The following result addresses the observability of the nonlinear discrete-time system (11).

Theorem 1: Suppose that the configuration of the Long

Baseline acoustic positioning system is such that

$$\mathbf{L} := \begin{bmatrix} 2(\mathbf{s}_1 - \mathbf{s}_2)^T & -\left(\|\mathbf{s}_1\|^2 - \|\mathbf{s}_2\|^2\right) \\ 2(\mathbf{s}_1 - \mathbf{s}_3)^T & -\left(\|\mathbf{s}_1\|^2 - \|\mathbf{s}_3\|^2\right) \\ \vdots & \vdots \\ 2(\mathbf{s}_{L-2} - \mathbf{s}_L)^T & -\left(\|\mathbf{s}_{L-2}\|^2 - \|\mathbf{s}_L\|^2\right) \\ 2(\mathbf{s}_{L-1} - \mathbf{s}_L)^T & -\left(\|\mathbf{s}_{L-1}\|^2 - \|\mathbf{s}_L\|^2\right) \end{bmatrix} \in \mathbb{R}^{C_2^L \times 4}$$

is full rank, i.e.,

$$\operatorname{rank}(\mathbf{L}) = 4. \tag{13}$$

Then, the discrete-time system (11) is observable on any interval $[k_i, k_{i+3}]$, $k_i = 0, 1, 2, \ldots$, in the sense that the initial state $\mathbf{x}(k_i)$ is uniquely determined by the input $\{\mathbf{u}(k) : k = k_i, k_{i+1}, k_{i+2}\}$ and the output $\{\mathbf{y}(k) : k = k_i, k_{i+1}, k_{i+2}\}$.

Proof: The proof resorts to Lemma 1 and it reduces to show that the observability matrix $\mathcal{O}(k_i, k_i+3)$ associated with the pair $(\mathbf{A}(k), \mathbf{C}(k))$ on $[k_i, k_{i+3}]$, $k_i > k_0$, has rank equal to the number of states of the system. Fix $k_i > k_0$ and suppose that the rank of the observability matrix is less than the number of states of the system. Then, there exists a unit vector $\mathbf{d} \in \mathbb{R}^{7+L+2}$, $\mathbf{d} = \begin{bmatrix} \mathbf{d}_1^T & \mathbf{d}_2^T & d_3 & \mathbf{d}_4^T & d_5 & d_6 \end{bmatrix}^T$, with $\mathbf{d}_1, \mathbf{d}_2 \in \mathbb{R}^3$, $d_3 \in \mathbb{R}$, $\mathbf{d}_4 \in \mathbb{R}^L$, $d_5, d_6 \in \mathbb{R}$, such that $\mathcal{O}(k_i, k_i+3)\,\mathbf{d} = \mathbf{0}$ or, equivalently,

$$\begin{cases} \mathbf{C}(k_i)\,\mathbf{d} = \mathbf{0} \\ \mathbf{C}(k_i+1)\,\mathbf{A}(k_i)\,\mathbf{d} = \mathbf{0} \\ \mathbf{C}(k_i+2)\,\mathbf{A}(k_i+1)\,\mathbf{A}(k_i)\,\mathbf{d} = \mathbf{0} \end{cases}. \tag{14}$$

From the first equation of (14), and attending to the structure of $\mathbf{C}(k_i)$, one immediately concludes that $\mathbf{d}_4 = \mathbf{0}$. Substituting that in the first equation of (14) gives

$$\begin{cases} 2(\mathbf{s}_1 - \mathbf{s}_2)^T \mathbf{d}_1 - \left(\|\mathbf{s}_1\|^2 - \|\mathbf{s}_2\|^2\right) d_3 = 0 \\ 2(\mathbf{s}_1 - \mathbf{s}_3)^T \mathbf{d}_1 - \left(\|\mathbf{s}_1\|^2 - \|\mathbf{s}_3\|^2\right) d_3 = 0 \\ \vdots \\ 2(\mathbf{s}_{L-2} - \mathbf{s}_L)^T \mathbf{d}_1 - \left(\|\mathbf{s}_{L-2}\|^2 - \|\mathbf{s}_L\|^2\right) d_3 = 0 \\ 2(\mathbf{s}_{L-1} - \mathbf{s}_L)^T \mathbf{d}_1 - \left(\|\mathbf{s}_{L-1}\|^2 - \|\mathbf{s}_L\|^2\right) d_3 = 0 \end{cases}. \tag{15}$$

Under Assumption (13), the only solution of (15) is $\mathbf{d}_1 = \mathbf{0}$ and $d_3 = 0$. Now, with $\mathbf{d}_1 = \mathbf{0}$, $d_3 = 0$, and $\mathbf{d}_4 = \mathbf{0}$, one may write, from the second equation of (14), that

$$\begin{cases} (\mathbf{s}_1 - \mathbf{s}_2)^T \mathbf{d}_2 = 0 \\ (\mathbf{s}_1 - \mathbf{s}_3)^T \mathbf{d}_2 = 0 \\ \vdots \\ (\mathbf{s}_{L-2} - \mathbf{s}_L)^T \mathbf{d}_2 = 0 \\ (\mathbf{s}_{L-1} - \mathbf{s}_L)^T \mathbf{d}_2 = 0 \end{cases}. \tag{16}$$

Again, under Assumption (13), the only solution of (16) is $\mathbf{d}_2 = \mathbf{0}$. Substituting that in the second equation of (14), together with $\mathbf{d}_1 = \mathbf{0}$, $d_3 = 0$, and $\mathbf{d}_4 = \mathbf{0}$ gives

$$2d_5 + Td_6 = 0. \tag{17}$$

Substituting $\mathbf{d}_1 = \mathbf{d}_2 = \mathbf{0}$, $d_3 = 0$, and $\mathbf{d}_4 = \mathbf{0}$ in the third equation of (14) allows to write

$$d_4 + Td_5 = 0. \tag{18}$$

The only solution of (17)-(18) is $d_5 = d_6 = 0$. But this contradicts the hypothesis of existence of a unit vector $\mathbf{d}$ such that (14) holds. Hence, the observability matrix must have rank equal to the number of states of the system. As the derivation remains unchanged for any other different $k_i > k_0$, the proof is concluded invoking Lemma 1. ■

Finally, it is important to stress that, in the definition of the augmented system (11), the original nonlinear outputs $r_i(k) = \sqrt{x_3(k+1)}\left\|\mathbf{s}_i - \frac{\mathbf{x}_1(k+1)}{x_3(k+1)}\right\|$, $i = 1, \ldots, L$, were discarded. Furthermore, there is nothing in (11) imposing the nonlinear constraints (8). While it is true that these restrictions could be easily imposed including artificial outputs, e.g., $x_{4+L}(k) - \mathbf{x}_1(k) \cdot \mathbf{x}_2(k) / x_3(k) = 0$, this form was preferred as it allows to apply Lemma 1. However, care must be taken when extrapolating conclusions from the observability of (11) to the observability of (5). The following theorem addresses this issue and provides the means for design of a state observer or filter for (5), as it will be seen shortly after.

Theorem 2: Suppose that (13) holds. Then:

i) the nonlinear system (5) is observable on any interval $[k_i, k_{i+3}]$, $k_i = 0, 1, 2, \ldots$, in the sense that the initial state $\mathbf{x}(k_i)$ is uniquely determined by the input $\{\mathbf{u}(k) : k = k_i, k_{i+1}, k_{i+2}\}$ and the output $\{r_1(k), \ldots, r_L(k) : k = k_i, k_{i+1}, k_{i+2}\}$; and
ii) the initial condition of the augmented nonlinear system (11) matches that of (5), i.e.,

$$\begin{cases} \mathbf{x}_1(k_i) = v_s^2(t_{k_i})\,\mathbf{p}(t_{k_i}) \\ \mathbf{x}_2(k_i) = v_s^2(t_{k_i})\,\mathbf{v}_c(t_{k_i}) \\ x_3(k_i) = v_s^2(t_{k_i}) \\ x_4(k_i) = v_s(t_{k_i})\,\|\mathbf{s}_1 - \mathbf{p}(t_{k_i})\| \\ \vdots \\ x_{3+L}(k_i) = v_s(t_{k_i})\,\|\mathbf{s}_L - \mathbf{p}(t_{k_i})\| \\ x_{4+L}(k_i) = v_s^2(t_{k_i})\,\mathbf{p}(t_{k_i}) \cdot \mathbf{v}_c(t_{k_i}) \\ x_{5+L}(k_i) = v_s^2(t_{k_i}) = \|\mathbf{v}_c(t_{k_i})\|^2 \end{cases}.$$

Proof: The second part of the theorem is established comparing the outputs of both systems as a function of their initial state. This is omitted due to the lack of space. Then, notice that, using Theorem 1, the initial condition of (11) is uniquely determined. Hence, it follows due to the correspondence between the two systems, that the initial condition of (5) is also uniquely determined. ■

C. Estimation solution

1) Augmented system: The means to design an observer for the quantities $v_s^2(t_k)\,\mathbf{p}(t_k)$, $v_s^2(t_k)\,\mathbf{v}_c(t_k)$, and $v_s^2(t_k)$ are provided by Theorem 2 as it is shown that an observer for (11), which can be regarded as linear for observer design purposes, suffices. A simple Kalman filter can be applied, yielding globally exponentially stable error dynamics if the system is shown to be uniformly completely observable [15]. In the paper, the pair $(\mathbf{A}(k), \mathbf{C}(k))$ was shown to be observable. The proof of uniform complete observability follows similar steps considering uniform bounds in time. It is omitted due to space limitations. An alternative to the Kalman filter could be the design of a Luenberger observer as detailed in [16, Theorem 29.2], which would allow to choose the convergence rate.

Notice that, even though the ocean current velocity and the factor that accounts for the unknown sound speed velocity are assumed constant, in nominal terms, by appropriate tuning of the Kalman filter it is possible to successfully track slowly time-varying quantities.

2) Estimation between range measurements: An observer (or filter) for the discrete-time system (11), as previously derived, only provides estimates when there are range measurements. However, the relative velocity and attitude measurements are usually available at a much higher rate than the range readings. As such, it is possible to obtain estimates of the scaled position, scaled velocity, and speed to propagation of the acoustic waves, at a higher rate, using open-loop propagation between range measurements, as given by

$$\begin{cases} \hat{\mathbf{x}}_1(t) = \hat{\mathbf{x}}_1(t_k) + (t - t_k)\,\hat{\mathbf{x}}_2(t_k) \\ \qquad\quad + \hat{x}_3(t_k) \int_{t_k}^{t} \mathbf{R}(\tau)\,\mathbf{v}_r(\tau)\, d\tau \\ \hat{\mathbf{x}}_2(t) = \hat{\mathbf{x}}_2(t_k) \\ \hat{x}_3(t) = \hat{x}_3(t_k) \end{cases}$$

for $t_k < t < t_{k+1}$.

3) Estimates of $\mathbf{p}(t)$*,* $\mathbf{v}_c(t)$*, and* $v_s(t)$*:* Estimates for $\mathbf{p}(t_k)$, $\mathbf{v}_c(t_k)$, and $v_s(t)$ follow from the Kalman filter or the Luenberger observer estimates, under some mild assumptions.

Assumption 1: The speed of propagation of the acoustic waves in the medium satisfies $V_m \leq v_s(t) \leq V_M$, with $V_m, V_M > 0$.

Assumption 2: The inertial position of the vehicle and the ocean current velocity are norm-bounded.

Considering estimates $\hat{x}_3(t)$ with globally exponentially stable error dynamics, the estimate of the speed of propagation of the acoustic waves in the medium can be obtained from

$$\hat{v}_s(t) = \begin{cases} V_m, & \hat{x}_3(t) < V_m^2 \\ \sqrt{\hat{x}_3(t)}, & V_m^2 < \hat{x}_3(t) < v_M^2 \\ V_M, & \hat{x}_3(t) > V_M^2 \end{cases},$$

whose error also converges globally exponentially fast to zero under Assumption 1. Estimates for the position and ocean current velocity then follow from

$$\begin{cases} \hat{\mathbf{p}}(t) = \frac{\hat{\mathbf{x}}_1(t)}{\hat{v}_s^2(t)} \\ \hat{\mathbf{v}}_c(t) = \frac{\hat{\mathbf{x}}_2(t)}{\hat{v}_s^2(t)} \end{cases},$$

and it is possible to show that, under Assumptions 1 and 2, these also follow globally exponentially stable error dynamics. These (rather trivial) results are omitted due to space constraints and are left for an extended version of the paper.

IV. Simulation results

This section presents a numerical simulation in order to exemplify the achievable performance with the proposed solution for long baseline navigation with explicit estimation of the velocity of propagation of the acoustic waves. These are only preliminary results and extensive Monte Carlo simulations will be carried out in the future, prior to experimental validation, as well as comparison with the Extended Kalman filter, which does not offer globally exponentially stable error dynamics.

The initial position of the vehicle is $\mathbf{p}(0) = [0\,0\,10]^T$ m, while the ocean current velocity was set to $\mathbf{v}_c(t) = [-0.1\,0.2\,0]^T$ m/s. The trajectory that was described by the vehicle is shown in Fig. 1. The LBL configuration is composed of 5 acoustic transponders and their inertial positions are $\mathbf{s}_1 = \begin{bmatrix} 0 & 0 & 1000 \end{bmatrix}^T$ (m), $\mathbf{s}_2 = \begin{bmatrix} 1000 & 0 & 500 \end{bmatrix}$ (m), $\mathbf{s}_3 = \begin{bmatrix} 0 & 750 & 500 \end{bmatrix}$ (m), $\mathbf{s}_4 = \begin{bmatrix} 0 & 0 & 500 \end{bmatrix}$ (m), and $\mathbf{s}_5 = \begin{bmatrix} 1000 & 1000 & 500 \end{bmatrix}$ (m), hence satisfying the rank condition (13). The velocity of propagation factor was set to $v_s(t) = 1.05$.

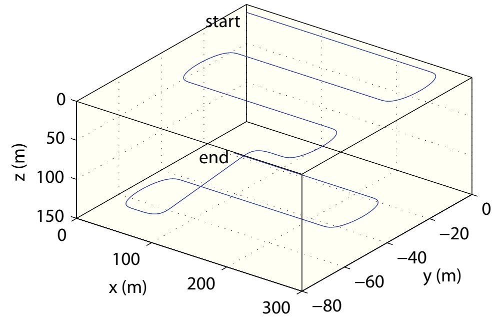


Fig. 1. Trajectory described by the vehicle

Sensor noise was considered for all sensors. In particular, the LBL range measurements and the DVL relative velocity readings were assumed to be corrupted by additive uncorrelated zero-mean white Gaussian noise, with standard deviations of $1\,\mathrm{m}$ and $0.01\,\mathrm{m/s}$, respectively. The attitude, provided by the AHRS and parameterized by roll, pitch, and yaw Euler angles, was also assumed to be corrupted by zero-mean, additive white Gaussian noise, with standard deviation of $0.03°$ for the roll and pitch and $0.3°$ for the yaw. The sampling period for the range measurements was set to $T = 1\,\mathrm{s}$, while the remaining sensors were sampled at 100 Hz. The discrete time input $\mathbf{u}(k)$, corresponding to a definite integral, was approximated using the trapezoid rule, while the open-loop solution of the position and ocean current velocity estimates, between range measurements, was computed using the Euler method. In fact, as it also corresponds to a definite integral, it is equivalent to the application of the trapezoid rule.

To tune the Kalman filter, the state disturbance covariance matrix was chosen as

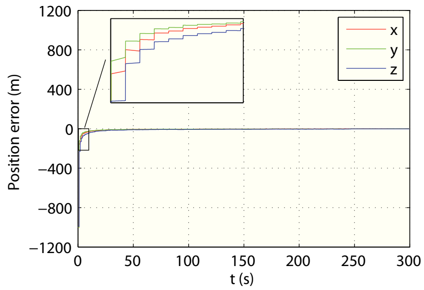
$$\textbf{blkdiag}\left(10^{-3}\mathbf{I},\, 10^{-5}\mathbf{I},\, 10^{-5},\, 10^{-2}\mathbf{I},\, 10^{-2},\, 10^{-5}\right)$$

and the output noise covariance matrix was set to

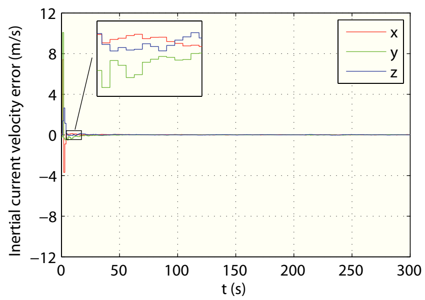
$$\textbf{blkdiag}\left(\mathbf{I},\, 0.5\mathbf{I}\right).$$

The initial condition for the position was set with a large initial error, $\hat{\mathbf{x}}_1(0) = [1000\,1000\,1000]^T$ (m), while the velocity of propagation factor estimate was set to $\hat{x}_3(0) = 1$. The states corresponding to the range measurements were set according to the initial range measurements and the remaining initial state estimates were set to zero.

The initial convergence of the position and velocity errors is depicted in Fig. 2, along with details of the discrete-time updates and open-loop propagation between range measurements, which translates into linearly increasing position errors between range measurements (approximately, due to noise). The detailed evolutions of the position and velocity errors are depicted in Figs. 3 and 4, respectively. The most noticeable feature is that the position and velocity errors remain, most of the time, below $1\,\mathrm{m}$ and $0.03\,\mathrm{m/s}$, respectively. The evolution of the error of the speed of propagation of the acoustic waves is shown in Fig. 5. The relevant feature here is that the error remains well below 0.5%.



(a) Position error



(b) Ocean current velocity error

Fig. 2. Initial convergence of the errors

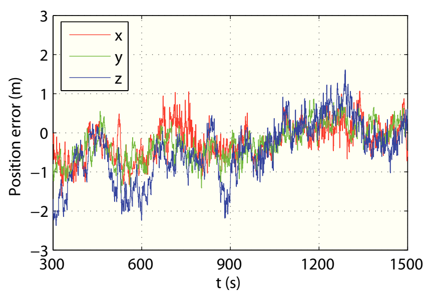


Fig. 3. Steady-state evolution of the position error

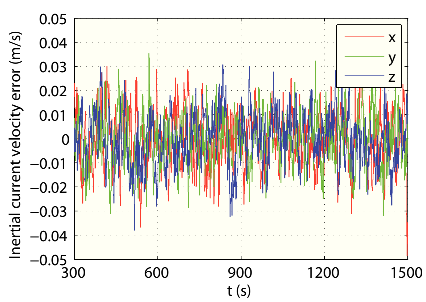


Fig. 4. Steady-state evolution of the ocean current velocity error

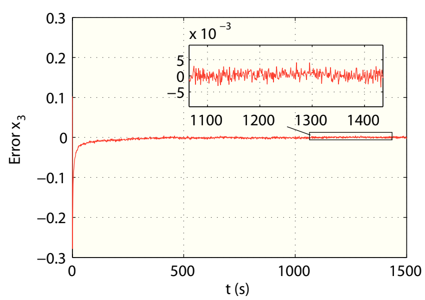


Fig. 5. Evolution of the error of $x_3\,(k)$

V. Conclusions

A common assumption in Long Baseline navigation is that speed of propagation of the acoustic waves in the medium is either known or measured. This paper presents a novel long baseline navigation framework where the factor related to the speed of propagation of the waves is explicitly taken into account and estimated. Considering discrete-time range measurements, an augmented system is proposed that can be regarded, for observability and observer design purposes, as linear. Its observability was analyzed and sufficient conditions were derived. The Kalman filter provides the estimation solution, with globally exponentially stable error dynamics, and DVL and AHRS measurements, obtained at higher rates, are integrated to obtain estimates at high rates. Simulation results evidence fast convergence and good performance.

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