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GES Source Localization based on Discrete-Time Position and Single Range Measurements

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Abstract— **This paper addresses the problem of estimating the position of a drifting source relative to an agent, in 3-D, based on discrete-time range measurements from the agent to the source, in addition to the position of the agent itself. An augmented nonlinear system is derived, in discrete-time, that can be regarded as linear for observability and observer design purposes. The analysis of the observability follows and sufficient conditions are derived, based directly on the trajectory of the agent. A Kalman filter with globally exponentially stable error dynamics is proposed and simulation results are presented that illustrate the achievable performance with the proposed solution.**

I. INTRODUCTION

A recurring problem that agents face in robotics is that of estimating the position of an external object (which may be another agent), either in absolute or relative coordinates. This problem is often denominated as that of source localization, when there is a device in the objects that emits signals (hence the designation as source), although sometimes it is also referred to as target localization, more often in warfare applications. In mobile robotics this can be the case of an unmanned ground vehicle attempting to estimate the position of a beacon, while in aerial robotics it can be an unmanned aerial vehicle trying to estimate the position of another unmanned aerial vehicle.

While a variety of sensors have been considered to solve this problem, such as direct positioning systems or the use of bearing measurements, see e.g. [1], [2], and [3], a lot of interest has sprouted recently in the use of range measurements from the agent to the source. In [4] a continuous-time localization algorithm for a fixed source is proposed, which achieves globally exponentially stable error dynamics under a persistent excitation condition. The analysis is further extended to the case of a non-stationary source, where it is shown that it is possible to achieve tracking up to some bounded error. In [5] the same problem is addressed considering a fixed source and discrete-time range measurements. A recursive least squares approach is proposed and the dependence of the covariance of the source position on the velocity profile of the vehicle is discussed. In [6] the problem of simultaneous estimation of the position of an autonomous underwater vehicle and the position of stationary range-only beacons was addressed, including an outlier rejection method that can identify groups of range measurements that are consistent with each other, and a method for initializing beacon positions in an extended Kalman filter (EKF).

In previous work by the authors, [7], the problem of source localization based on single range measurements and relative velocity readings was addressed for a fixed source, in a continuous-time framework. This paper builds on those results considering instead a drifting source, discrete-time range measurements, and that the agent has access to its position instead of relative velocity readings. A discrete-time nonlinear augmented system is derived and its observability analyzed, in a constructive manner, such that the design of an observer (or filter) follows naturally using estimation tools for linear systems, even though the system is linear. A Kalman filter is proposed with globally exponentially stable error dynamics.

The paper is organized as follows. The problem statement and the nominal system dynamics are introduced in Section II, while the filter design is detailed in Section III. Simulation results are presented in Section IV and Section V summarizes the main results of the paper.

A. Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. A block diagonal matrix is written as $\mathbf{blkdiag}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$. For $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \cdot \mathbf{y}$ represents the inner product.

II. PROBLEM STATEMENT

Consider an agent, whose inertial position at time $t = t_k$, $t_k = t_0 + kT$, $k = 1, 2, \ldots, T > 0$, is denoted as $\mathbf{p}(k) \in \mathbb{R}^3$, seeking to determine the position of a source, whose inertial position at time $t = t_k$ is denoted as $\mathbf{s}(k) \in \mathbb{R}^3$. Assume that the source is drifting with constant velocity $\mathbf{v}(k) \in \mathbb{R}^3$. Finally, suppose that the agent has access to its own position, $\mathbf{p}(k)$, and measures its distance to the source, given by

$$r(k) = \|\mathbf{s}(k) - \mathbf{p}(k)\| \in \mathbb{R}.$$

The problem considered in this paper is that of estimating the position of the source, $\mathbf{s}(k)$, as well as its drifting velocity, $\mathbf{v}(k)$, based on the agent position measurements, $\mathbf{p}(k)$, and the range measurements, $r(k)$.

The evolution of the position of the source is simply given by $\mathbf{s}(k+1) = \mathbf{s}(k) + T\mathbf{v}(k)$, where $T > 0$ denotes the sampling period. On the other hand, as the source

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drifting velocity is assumed constant, its evolution is simply described by $\mathbf{v}(k+1) = \mathbf{v}(k)$. Hence, the discrete-time system dynamics are given by

$$\left\{\begin{array}{l}\mathbf{s}(k+1) = \mathbf{s}(k) + T\mathbf{v}(k) \\ \mathbf{v}(k+1) = \mathbf{v}(k) \\ r(k+1) = \|\mathbf{s}(k+1) - \mathbf{p}(k+1)\|\end{array}\right. . \quad (1)$$

In other words, the problem considered herein is the design of an estimator for (1) with globally exponentially stable error dynamics.

III. Filter design

In previous work by the authors [7] the problem of source localization based on single range measurements was addressed considering a continuous framework and, in addition to the distance, relative velocity measurements. In the proposed approach, the range measurement was considered as a system state and additional states were defined identifying the resulting nonlinear terms of the dynamics of the range measurement. The present paper follows up on previous work considering: i) discrete-time measurements, as opposed to continuous signals; ii) a drifting source, instead of a fixed one; and iii) that the agent has access to its own position instead of relative velocity readings. Naturally, instead of localizing the source relative to the agent, the solution proposed herein provides the absolute position of the source.

A. System augmentation

Define as discrete-time states

$$\left\{\begin{array}{l}\mathbf{x}_1(k) := \mathbf{s}(k) \\ \mathbf{x}_2(k) := \mathbf{v}(k)\end{array}\right.$$

and consider the range measurement as an additional system state, i.e., define

$$x_3(k) := r(k).$$

In order to describe the evolution of $x_3(k)$, consider the square of the distance and expand

$$r^2(k+1) = \|\mathbf{x}_1(k+1) - \mathbf{p}(k+1)\|^2$$

using (1), which gives

$$\begin{aligned}r^2(k+1) = & -2[\mathbf{p}(k+1) - \mathbf{p}(k)] \cdot \mathbf{x}_1(k) \\ & -2T\mathbf{p}(k+1) \cdot \mathbf{x}_2(k) \\ & + r^2(k) + 2T\mathbf{x}_1(k) \cdot \mathbf{x}_2(k) + T^2\|\mathbf{x}_2(k)\|^2 \\ & + \|\mathbf{p}(k+1)\|^2 - \|\mathbf{p}(k)\|^2 . \quad (2)\end{aligned}$$

Identifying the nonlinear terms $\mathbf{x}_1(k) \cdot \mathbf{x}_2(k)$ and $\|\mathbf{x}_2(k)\|^2$ in (2) with new system states, i.e.,

$$\left\{\begin{array}{l}x_4(k) := \mathbf{x}_1(k) \cdot \mathbf{x}_2(k) \\ x_5(k) := \|\mathbf{x}_2(k)\|^2\end{array}\right. , \quad (3)$$

and noticing that $r^2(k) = x_3(k)r(k)$ allows to rewrite (2)

$$\begin{aligned}x_3(k+1) = & -2\frac{[\mathbf{p}(k+1) - \mathbf{p}(k)] \cdot \mathbf{x}_1(k)}{r(k+1)} \\ & -2T\frac{\mathbf{p}(k+1) \cdot \mathbf{x}_2(k)}{r(k+1)} \\ & + \frac{r(k)x_3(k)}{r(k+1)} + \frac{2Tx_4(k)}{r(k+1)} \\ & + \frac{T^2x_5(k)}{r(k+1)} + \frac{\|\mathbf{p}(k+1)\|^2 - \|\mathbf{p}(k)\|^2}{r(k+1)} . \quad (4)\end{aligned}$$

The evolution of the new states can be simply written, using (1), as

$$\left\{\begin{array}{l}x_4(k+1) = x_4(k) + Tx_5(k) \\ x_5(k+1) = x_5(k)\end{array}\right. .$$

Define the augmented state vector as

$$\mathbf{x}(k) := \begin{bmatrix}\mathbf{x}_1(k) \\ \mathbf{x}_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k)\end{bmatrix} \in \mathbb{R}^{3+3+3}.$$

Then, the discrete-time system dynamics can be written as

$$\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}u(k),$$

where

$$\mathbf{A}(k) = \begin{bmatrix}\mathbf{I} & T\mathbf{I} & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & 0 & 0 \\ & & \mathbf{A}_3(k) & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 1\end{bmatrix} \in \mathbb{R}^{9\times 9},$$

with

$$\mathbf{A}_3(k) = \begin{bmatrix}-2\frac{\mathbf{p}(k+1) - \mathbf{p}(k)}{r(k+1)} \\ -2T\frac{\mathbf{p}(k+1)}{r(k+1)} \\ \frac{r(k)}{r(k+1)} \\ \frac{2T}{r(k+1)} \\ \frac{T^2}{r(k+1)}\end{bmatrix}^T,$$

$$\mathbf{B} = \begin{bmatrix}\mathbf{0} \\ \mathbf{0} \\ 1 \\ 0 \\ 0\end{bmatrix} \in \mathbb{R}^{9\times 1},$$

and

$$u(k) = \frac{\|\mathbf{p}(k+1)\|^2 - \|\mathbf{p}(k)\|^2}{r(k+1)}.$$

Discarding the original nonlinear system output, one may consider as a new system output $y(k+1) = x_3(k+1)$. The discrete-time augmented system can then be written as

$$\left\{\begin{array}{l}\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}u(k) \\ y(k+1) = \mathbf{C}\mathbf{x}(k+1)\end{array}\right. , \quad (5)$$

where $\mathbf{C} := \begin{bmatrix}\mathbf{0} & \mathbf{0} & 1 & 0 & 0\end{bmatrix}$.

Remark 1: By definition, the distance is nonnegative. In addition, it is impossible to have a null distance as that would imply that the agent was on top of the source, which is physically impossible. In fact, there is always a minimum distance to the transponders. Hence, (5) is well defined.

B. Observability analysis

The system (5) can be regarded as a discrete linear time-varying system for observer design purposes, even though the system matrix $\mathbf{A}(k)$ depends explicitly on the range measurement, which is both a state and an output. This is possible because for observer (or filter) design purposes the range is available and hence it can be simply considered as a function of time. This idea was first pursued by the authors in [7, Lemma 1] for a similar continuous system, whose application is equivalent for the discrete-time case, as shown in the following theorem.

Lemma 1: Consider the nonlinear discrete-time system

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{\mathcal{A}}\big(k,\boldsymbol{\mathcal{U}}_{k_0}^{k+1},\boldsymbol{\mathcal{Y}}_{k_0}^{k+1}\big)\mathbf{x}(k) + \boldsymbol{\mathcal{B}}\big(k,\boldsymbol{\mathcal{U}}_{k_0}^{k+1},\boldsymbol{\mathcal{Y}}_{k_0}^{k+1}\big)\boldsymbol{u}(k) \\ \boldsymbol{y}(k+1) = \boldsymbol{\mathcal{C}}\left(k,\boldsymbol{\mathcal{U}}_{k_0}^{k+1},\boldsymbol{\mathcal{Y}}_{k_0}^{k+1}\right)\boldsymbol{x}(k+1) \end{cases}, \tag{6}$$

where $\boldsymbol{\mathcal{U}}_{k_0}^{k_f} := \{\boldsymbol{u}(k_0), \mathbf{u}(k_0+1), \ldots, \mathbf{u}(k_f-1)\}$ and $\boldsymbol{\mathcal{Y}}_{k_0}^{k_f} := \{\boldsymbol{y}(k_0), \mathbf{y}(k_0+1), \ldots, \mathbf{y}(k_f-1)\}$ are the input and output signals, respectively, on the time interval $[k_0, k_f[$, and $\boldsymbol{x}(k) \in \mathbb{R}^n$. If $\mathrm{rank}\left(\boldsymbol{\mathcal{O}}\left(\mathrm{k_0}, \mathrm{k_f}\right)\right) = \mathrm{n}$, where $\boldsymbol{\mathcal{O}}(k_0, k_f)$ is the observability matrix associated with the pair $\left(\boldsymbol{\mathcal{A}}\left(k,\boldsymbol{\mathcal{U}}_{k_0}^{k_f},\boldsymbol{\mathcal{Y}}_{k_0}^{k_f}\right),\boldsymbol{\mathcal{C}}\left(k,\boldsymbol{\mathcal{U}}_{k_0}^{k_f},\boldsymbol{\mathcal{Y}}_{k_0}^{k_f}\right)\right)$ on $\mathcal{I} := [k_0, k_f]$, then the nonlinear system (6) is observable on $\mathcal{I}$ in the sense that, given the system input and output signals $\boldsymbol{\mathcal{U}}_{k_0}^{k_f}$ and $\boldsymbol{\mathcal{Y}}_{k_0}^{k_f}$, the initial condition $\boldsymbol{x}(k_0)$ is uniquely defined.

Proof: For the sake of ease of notation, and as both the system input and output signals $\boldsymbol{\mathcal{U}}_{k_0}^{k_f}$ and $\boldsymbol{\mathcal{Y}}_{k_0}^{k_f}$ are assumed available, consider the simplified notation $\boldsymbol{\mathcal{A}}(k) = \boldsymbol{\mathcal{A}}\big(k,\boldsymbol{\mathcal{U}}_{k_0}^{k+1},\boldsymbol{\mathcal{Y}}_{k_0}^{k+1}\big)$, $\boldsymbol{\mathcal{B}}(k) = \boldsymbol{\mathcal{B}}\big(k,\boldsymbol{\mathcal{U}}_{k_0}^{k+1},\boldsymbol{\mathcal{Y}}_{k_0}^{k+1}\big)$, and $\boldsymbol{\mathcal{C}}(k) = \boldsymbol{\mathcal{C}}\big(k,\boldsymbol{\mathcal{U}}_{k_0}^{k+1},\boldsymbol{\mathcal{Y}}_{k_0}^{k+1}\big)$. Given $\boldsymbol{\mathcal{U}}_{k_0}^{k_f}$ and $\boldsymbol{\mathcal{Y}}_{k_0}^{k_f}$, it is possible to compute the transition matrix associated with the system matrix $\boldsymbol{\mathcal{A}}(k)$, given by $\boldsymbol{\phi}(k, k_0) = \boldsymbol{\mathcal{A}}(k-1)\boldsymbol{\mathcal{A}}(k-2)\ldots\boldsymbol{\mathcal{A}}(k_0)$ for $k_0 < k \leq k_f$, with $\boldsymbol{\phi}(k_0, k_0) = \mathbf{I}$. Hence, it is possible to compute the observability matrix

$$\boldsymbol{\mathcal{O}}(k_0, k_f) = \begin{bmatrix} \boldsymbol{\mathcal{C}}(k_0) \\ \boldsymbol{\mathcal{C}}(k_0+1)\boldsymbol{\phi}(k_0+1, k_0) \\ \vdots \\ \boldsymbol{\mathcal{C}}(k_f-1)\boldsymbol{\phi}(k_f-1, k_0) \end{bmatrix}.$$

Now, notice that it is possible to write the evolution of the state, given the system input and output (which allow to compute the transition matrix), as

$$\boldsymbol{x}(k) = \boldsymbol{\phi}(k, k_0)\boldsymbol{x}_0 + \sum_{j=k_0}^{k-1}\boldsymbol{\phi}(k, j+1)\boldsymbol{\mathcal{B}}(j)\boldsymbol{u}(j) \tag{7}$$

for $k_0 < k < k_f$, where $\boldsymbol{x}_0 = \boldsymbol{x}(k_0)$ is the initial condition. This is easily verified by substitution into the state equation. The remainder of the proof follows as in classic theory. The output of the system can be written, from (7), as

$$\boldsymbol{y}(k) = \boldsymbol{\mathcal{C}}(k)\boldsymbol{\phi}(k, k_0)\boldsymbol{x}_0 + \boldsymbol{\mathcal{C}}(k)\sum_{j=k_0}^{k-1}\boldsymbol{\phi}(k, j+1)\boldsymbol{\mathcal{B}}(j)\boldsymbol{u}(j)$$

for $k_0 < k < k_f$, with $\boldsymbol{y}(k_0) = \boldsymbol{\mathcal{C}}(k_0)\boldsymbol{x}_0$. Considering the output for all available time instants gives

$$\begin{bmatrix} \boldsymbol{y}(k_0) \\ \boldsymbol{y}(k_0+1) \\ \boldsymbol{y}(k_0+2) \\ \vdots \\ \boldsymbol{y}(k_f-1) \end{bmatrix} = \boldsymbol{\mathcal{O}}(k_0, k_f)\boldsymbol{x}_0 + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\mathcal{C}}(k_0+1)\boldsymbol{\mathcal{B}}(k_0)\boldsymbol{u}(k_0) \\ \boldsymbol{\mathcal{C}}(k_0+2)\sum_{j=k_0}^{k_0+1}\boldsymbol{\phi}(k_0+2, j+1)\boldsymbol{\mathcal{B}}(j)\boldsymbol{u}(j) \\ \vdots \\ \boldsymbol{\mathcal{C}}(k_f-1)\sum_{j=k_0}^{k_f-2}\boldsymbol{\phi}(k_f-1, j+1)\boldsymbol{\mathcal{B}}(j)\boldsymbol{u}(j) \end{bmatrix}. \tag{8}$$

Multiplying (8) on both sides by $\boldsymbol{\mathcal{O}}^T(k_0, k_f)$ and rearranging the terms yields

$$\boldsymbol{\mathcal{W}}(k_0, k_f)\boldsymbol{x}_0 = \boldsymbol{\mathcal{O}}^T(k_0, k_f)\begin{bmatrix} \boldsymbol{y}(k_0) \\ \boldsymbol{y}(k_0+1) \\ \boldsymbol{y}(k_0+2) \\ \vdots \\ \boldsymbol{y}(k_f-1) \end{bmatrix} - \boldsymbol{\mathcal{O}}^T(k_0, k_f)\begin{bmatrix} \mathbf{0} \\ \boldsymbol{\mathcal{C}}(k_0+1)\boldsymbol{\mathcal{B}}(k_0)\boldsymbol{u}(k_0) \\ \boldsymbol{\mathcal{C}}(k_0+2)\sum_{j=k_0}^{k_0+1}\boldsymbol{\phi}(k_0+2, j+1)\boldsymbol{\mathcal{B}}(j)\boldsymbol{u}(j) \\ \vdots \\ \boldsymbol{\mathcal{C}}(k_f-1)\sum_{j=k_0}^{k_f-2}\boldsymbol{\phi}(k_f-1, j+1)\boldsymbol{\mathcal{B}}(j)\boldsymbol{u}(j) \end{bmatrix}, \tag{9}$$

where $\boldsymbol{\mathcal{W}}(k_0, k_f) := \boldsymbol{\mathcal{O}}^T(k_0, k_f)\,\boldsymbol{\mathcal{O}}(k_0, k_f)$ is the observability Gramian associated with the pair $(\boldsymbol{\mathcal{A}}(k),\boldsymbol{\mathcal{C}}(k))$ on $\mathcal{I}$. All quantities in (9) but $\boldsymbol{x}_0$ are known given the system input and output and as such (9) is a linear algebraic equation on $\boldsymbol{x}_0$. Hence, if $\mathrm{rank}\left(\boldsymbol{\mathcal{O}}\left(\mathrm{k_0}, \mathrm{k_f}\right)\right) = \mathrm{n}$, the observability Gramian $\boldsymbol{\mathcal{W}}(k_0, k_f)$ is invertible and therefore $\boldsymbol{x}_0$ is uniquely defined, concluding the proof. ■

Remark 2: It is important to remark that, even though (7) resembles, at first glance, the response of a linear system, that is not the case and it does not correspond to the superposition of the free response (due to the initial condition) and the forced response (due to system input). This is so because the transition matrix in (7) depends explicitly on the system input and output. However, that is not a problem for observability and observer design purposes as both the input and output signals are assumed available.

The following result addresses the observability of the nonlinear discrete-time system (5).

Theorem 1: If $\mathrm{rank}\left(\mathbf{M}_{\mathrm{N-1}}\right) = 8$, where

$$\mathbf{M}_i := \begin{bmatrix} [\mathbf{p}(k_0+1) - \mathbf{p}(k_0)]^T & \mathbf{p}^T(k_0+1) & 1 & 1^2 \\ [\mathbf{p}(k_0+2) - \mathbf{p}(k_0)]^T & 2\mathbf{p}^T(k_0+2) & 2 & 2^2 \\ [\mathbf{p}(k_0+3) - \mathbf{p}(k_0)]^T & 3\mathbf{p}^T(k_0+3) & 3 & 3^2 \\ \vdots & \vdots & \vdots & \vdots \\ [\mathbf{p}(k_0+i) - \mathbf{p}(k_0)]^T & i\mathbf{p}^T(k_0+i) & i & i^2 \end{bmatrix},$$

then (5) is observable on the interval $[k_0, k_0+N]$, in the sense that the initial state $\mathbf{x}(k_0)$ is uniquely determined by the input $\{\mathbf{u}(k), k = k_0, k_0+1, \ldots, k_0+N-1\}$ and the output $\{\mathbf{y}(k), k = k_0, k_0+1, \ldots, k_0+N-1\}$.

Proof: The proof amounts to show that the observability matrix $\mathcal{O}(k_0, k_0+N)$ associated with the pair $(\mathbf{A}(k), \mathbf{C})$ on $[k_0, k_0+N]$ has rank equal to the number of states of the system, i.e. rank 9, which immediately gives the desired result by application of Lemma 1. Suppose that the rank of the observability matrix is less than 9. Then, there exists a unit vector $\mathbf{d} \in \mathbb{R}^9$, $\mathbf{d} = \begin{bmatrix} \mathbf{d}_1^T & \mathbf{d}_2^T & d_3 & d_4 & d_5 \end{bmatrix}^T$, with $\mathbf{d}_1, \mathbf{d}_2 \in \mathbb{R}^3$, $d_3, d_4, d_5 \in \mathbb{R}$, such that $\mathcal{O}(k_0, k_0+N)\mathbf{d} = \mathbf{0}$ or, equivalently,

$$\begin{cases} \mathbf{C}\mathbf{d} = 0 \\ \mathbf{C}\mathbf{A}(k_0)\mathbf{d} = 0 \\ \mathbf{C}\mathbf{A}(k_0+1)\mathbf{A}(k_0)\mathbf{d} = 0 \\ \vdots \\ \mathbf{C}\mathbf{A}(k_0+N-2)\ldots\mathbf{A}(k_0+1)\mathbf{A}(k_0)\mathbf{d} = 0 \end{cases}. \quad (10)$$

Expanding the first equation of (10) implies that it must be $d_3 = 0$. Substituting that in the i-th equation of (10), $i > 1$, gives

$$-2\left[\mathbf{p}(k_0+i-1) - \mathbf{p}(k_0)\right]\cdot\mathbf{d}_1 - 2T\mathbf{p}(k_0+i-1)\cdot\mathbf{d}_2 \\ +2(i-1)Td_4 + (i-1)^2 T^2 d_5 = 0. \quad (11)$$

Now, let $\mathbf{d}' = \begin{bmatrix} -\frac{1}{2}\mathbf{d}_1^T & -\frac{1}{2T}\mathbf{d}_2^T & \frac{1}{2T}d_4 & \frac{1}{T^2}d_5 \end{bmatrix}^T$. Using (11) it is a simple matter of computation to show that $\mathbf{M}_{N-1}\mathbf{d}' = \mathbf{0}$, which means that $\operatorname{rank}(\mathbf{M}_{N-1}) < 8$, thus contradicting the hypothesis of the theorem. Hence, if $\operatorname{rank}(\mathbf{M}_{N-1}) = 8$, there cannot exist a unit vector $\mathbf{d}$ such that $\mathcal{O}(k_0, k_0+N)\mathbf{d} = \mathbf{0}$, which implies that the observability matrix must have rank equal to the number of states of the system, which concludes the proof invoking Lemma 1. ■

Notice that $r(k) = \|\mathbf{s}(k) - \mathbf{x}_1(k)\|$ was discarded in (5). Furthermore, there is nothing in (5) imposing the nonlinear constraints (3). While it is true that these restrictions could be easily imposed including artificial outputs, e.g., $x_4(k) - \mathbf{x}_1(k)\cdot\mathbf{x}_2(k) = 0$, this form was preferred as it allows to apply Lemma 1. However, care must be taken when extrapolating conclusions from the observability of (5) to the observability of (1). The following theorem addresses this issue and provides the means for design of a state observer or filter for (1).

Theorem 2: Under the conditions of Theorem 1, the nonlinear system (1) is observable on interval $[k_0, k_0+N]$ in the sense that the initial state $\mathbf{x}(k_0)$ is uniquely determined by the input $\{\mathbf{u}(k), k = k_0, k_0+1, \ldots, k_0+N-1\}$ and the output $\{\mathbf{y}(k), k = k_0, k_0+1, \ldots, k_0+N-1\}$. Furthermore, the initial condition of the augmented nonlinear system (5) matches that of (1) and an observer with globally exponentially stable error dynamics for (5) is also an observer for (1) with globally exponentially stable error dynamics.

Proof: Let

$$\mathbf{x}(k_0) := \begin{bmatrix} \mathbf{x}_1(k_0) \\ \mathbf{x}_2(k_0) \\ x_3(k_0) \\ x_4(k_0) \\ x_5(k_0) \end{bmatrix} \in \mathbb{R}^9$$

be the initial condition of (5), which, from Theorem 1, is uniquely determined, and let $\mathbf{s}(k_0)$ and $\mathbf{v}(k_0)$ be the initial condition of (1). From the output of both systems for $k = k_0$ one immediately concludes that it must be

$$x_3(k_0) = \|\mathbf{s}(k_0) - \mathbf{p}(k_0)\| = r(k_0). \quad (12)$$

Considering the output of the nonlinear system (1) for $k = k_0 + i$ as a function of its initial state allows to write, after some trivial computations,

$$\begin{aligned} r^2(k_0+i) = & -2\left[\mathbf{p}(k_0+i) - \mathbf{p}(k_0)\right]\cdot\mathbf{s}(k_0) \\ & -2iT\mathbf{p}(k_0+i)\cdot\mathbf{v}(k_0) \\ & +r^2(k_0) + 2iT\mathbf{s}(k_0)\cdot\mathbf{v}(k_0) + i^2T^2\|\mathbf{v}(k_0)\|^2 \\ & \|\mathbf{p}(k_0+i)\|^2 - \|\mathbf{p}(k_0)\|^2 \end{aligned} \quad (13)$$

for all $1 \le i \le N-1$. On the other hand, evaluating the output of (5) as a function of $\mathbf{x}(k_0)$ and using the fact that, by contruction, $x_3(k) = r(k)$, $k = k_0, k_0+1, \ldots$, allows to write, after some computations,

$$\begin{aligned} r^2(k_0+i) = & -2\left[\mathbf{p}(k_0+i) - \mathbf{p}(k_0)\right]\cdot\mathbf{x}_1(k_0) \\ & -2iT\mathbf{p}(k_0+i)\cdot\mathbf{x}_2(k_0) \\ & +r^2(k_0) + 2iTx_4(k_0) + i^2T^2x_5(k_0) \\ & \|\mathbf{p}(k_0+i)\|^2 - \|\mathbf{p}(k_0)\|^2 \end{aligned} \quad (14)$$

for all $1 \le i \le N-1$. Comparison between (13) and (14) gives

$$\begin{aligned} & -2\left[\mathbf{p}(k_0+i) - \mathbf{p}(k_0)\right]\cdot\left[\mathbf{s}(k_0) - \mathbf{x}_1(k_0)\right] \\ & -2iT\mathbf{p}(k_0+i)\cdot\left[\mathbf{v}(k_0) - \mathbf{x}_2(k_0)\right] \\ & +2iT\left[\mathbf{s}(k_0)\cdot\mathbf{v}(k_0) - x_4(k_0)\right] \\ & +i^2T^2\left[\|\mathbf{v}(k_0)\|^2 - x_5(k_0)\right] = 0 \end{aligned}$$

for all $1 \le i \le N-1$ or, equivalently,

$$\mathbf{M}_{N-1}\begin{bmatrix} -2\left[\mathbf{s}(k_0) - \mathbf{x}_1(k_0)\right] \\ -2T\left[\mathbf{v}(k_0) - \mathbf{x}_2(k_0)\right] \\ 2T\left[\mathbf{s}(k_0)\cdot\mathbf{v}(k_0) - x_4(k_0)\right] \\ T^2\left[\|\mathbf{v}(k_0)\|^2 - x_5(k_0)\right] \end{bmatrix} = \mathbf{0}. \quad (15)$$

Now, under the hypothesis that $\operatorname{rank}(\mathbf{M}_{N-1}) = 8$, the only solution of (15) is

$$\begin{cases} \mathbf{x}_1(k_0) = \mathbf{s}(k_0) \\ \mathbf{x}_2(k_0) = \mathbf{v}(k_0) \\ x_4(k_0) = \mathbf{s}(k_0)\cdot\mathbf{v}(k_0) \\ x_5(k_0) = \|\mathbf{v}(k_0)\|^2 \end{cases}.$$

This concludes the proof: i) it has been shown that the initial condition of (1) matches that of (5), which is uniquely determined as shown in Theorem 1, hence concluding the proof of the first part of the theorem; and ii) the second part of the theorem follows from the first: the estimation error of an observer for (5) with globally exponentially stable error dynamics converges to zero exponentially fast, which means that its estimate approaches the true state exponentially fast. But as the true state of (5) matches that of the nonlinear system (1), that means that an observer (filter) for (5) is also an observer (filter) for the original nonlinear system, with globally exponentially stable error dynamics. ■

C. Kalman filter and further discussion

The design of an observer solution for (5) can be done using a variety of tools for linear systems. Indeed, while (5) is in fact a nonlinear system, it has been shown that it can be regarded, for observability and observer design purposes, a linear in the state, as both the input and output signals are available. One option would be to design a Luenberger observer as detailed in [8, Theorem 29.2], which would allow to choose the convergence rate. In this paper, the celebrated Kalman filter is employed instead.

While for linear time invariant systems observability suffices to establish a Kalman filter with globally exponentially stable error dynamics, for linear time-varying systems stronger forms are required, in particular, uniform complete observability. Conditions for uniform complete observability of the pair $(\mathbf{A}(k), \mathbf{C})$ will be derived in future work, ensuring that the Kalman filter has globally exponentially stable error dynamics [9].

It is important to stress that, in spite of the fact that, in nominal terms, the velocity of the source was assumed constant, it is possible to consider, during the design of the Kalman filter, that this state is driven by a white Gaussian process, with zero mean. By appropriate adjustment of the magnitude of the corresponding filter parameter (state disturbance variance), it is possible to allow the filter to estimate slowly time-varying source velocities.

Finally, notice that there exists multiplicative noise, as the system matrices are noisy. Hence, no optimal claims are made in the paper. The design of the Kalman filter that is presented in the next section does not take this aspect into account.

IV. Simulation results

This section presents simulation results in order to give an idea of the attainable performance with the proposed solution. Nevertheless, these are only preliminary results and extensive Monte Carlo simulations should be carried out in the future, prior to experimental validation, to further validate the solution, as well as comparison with the Extended Kalman filter. The example that is presented here is inspired by that offered in [5], with the necessary adaptations as in the present approach the source is allowed to drift.

The initial position of the source is $\mathbf{s}(0) = [30\,0\,0]^T$ m, while its drift velocity was set to $\mathbf{v}(t) = [1\,0\,0]^T$ m/s. The trajectory of the agent is given by

$$\mathbf{p}(t) = \begin{bmatrix} t + 10\sin\left(\frac{1}{100}2\pi t\right) \\ 10\sin\left(\frac{2}{100}2\pi t\right) \\ 10\sin\left(\frac{3}{100}2\pi t\right) \end{bmatrix} \text{ m/s},$$

which provides sufficient excitation such that the system is uniformly completely observable. The resulting trajectory of the source relative to the agent is as depicted in Fig. 1.

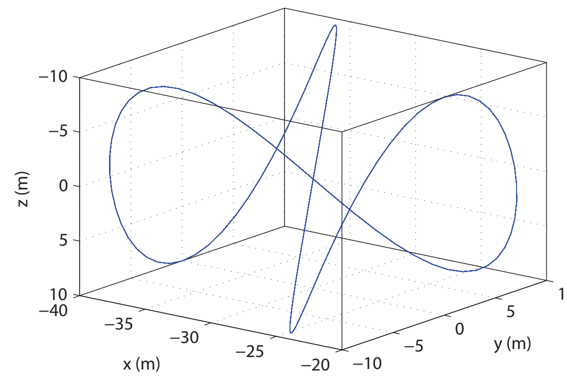


Fig. 1. Trajectory of the source relative to the agent, $\mathbf{s}(t) - \mathbf{p}(t)$

A sampling period of $T = 1$ s was employed in the simulations and sensor noise was considered for all sensors. In particular, the range measurements and agent position readings were assumed to be corrupted by additive uncorrelated zero-mean white Gaussian noise, with standard deviations of $0.3\,\mathrm{m}$ and $1\,\mathrm{m}$, respectively. To tune the Kalman filter, the state disturbance covariance matrix was chosen as

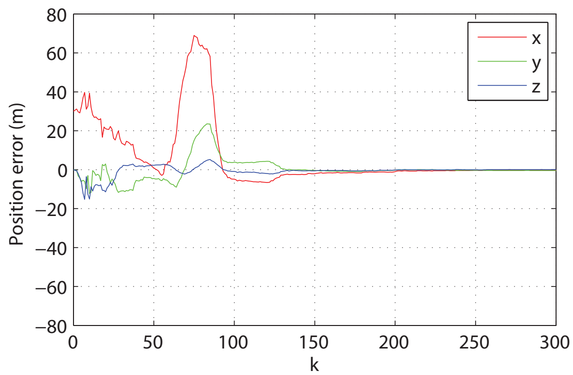
$$\textbf{blkdiag}\left(10^{-10}\mathbf{I},\, 10^{-10}\mathbf{I},\, 10^{-8},\, 10^{-10},\, 10^{-10}\right)$$

and the output noise variance was set to $0.3^2\,\mathrm{m/s}^2$. The initial condition is zero for all states except for the range, which was initialized according to the first range measurement.

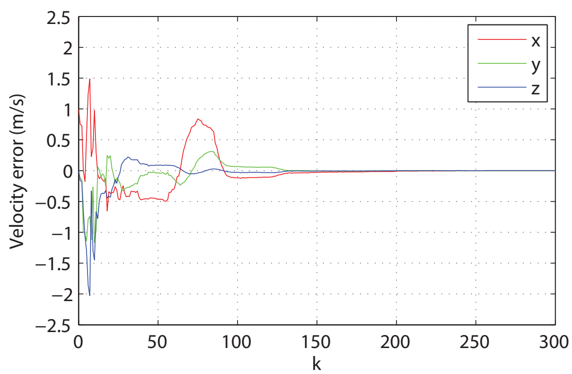
The initial convergence of the position and velocity errors is depicted in Fig. 2, whereas the initial evolution of the range errors is shown in Fig. 3. The convergence of the remaining state errors is shown in Fig. 4. As it can be seen from the various plots, the convergence rate of the filter is quite high. The detailed evolutions of the position and velocity errors are depicted in Figs. 5 and 6, respectively. The most noticeable feature is that the position and velocity errors remain, in steady-state, below $0.4\,\mathrm{m}$ and $0.002\,\mathrm{m/s}$, respectively.

V. Conclusions

This paper addressed the problem of source localization based on single range measurements, considering a discrete-time framework and a drifting source, with unknown constant velocity. Based on previous work by the authors, a discrete-time augmented nonlinear system was derived that can be regarded, for the purpose of state estimation, as linear. Its observability was studied and a Kalman filter provides the estimation solution, with globally exponentially stable error dynamics. Simulation results evidence fast convergence and good performance in the presence of sensor noise. Future work will cover the comparison with existing techniques, in particular with the Extended Kalman Filter (EKF), which does not offer GAS guarantees, and experimental evaluation.



(a) Position error



(b) Velocity error

Fig. 2. Initial convergence of error

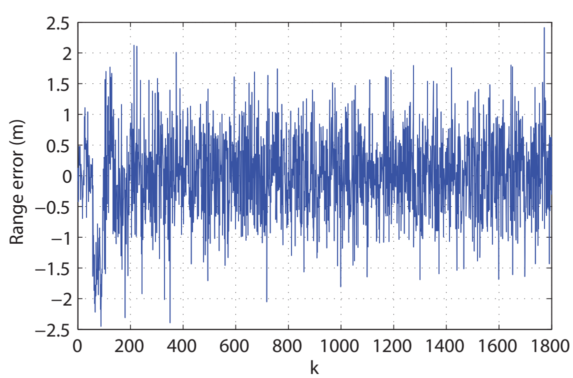


Fig. 3. Initial convergence of the range error

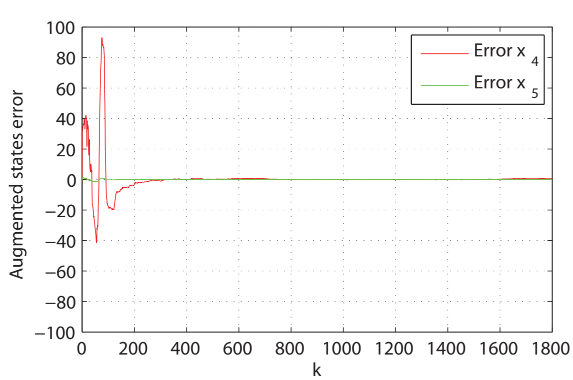


Fig. 4. Evolution of the error of the augmented states $x_4(k)$ and $x_5(k)$, in red and green, respectively

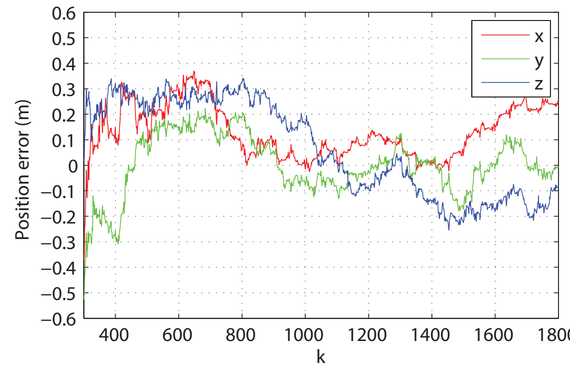


Fig. 5. Steady-state evolution of the position error

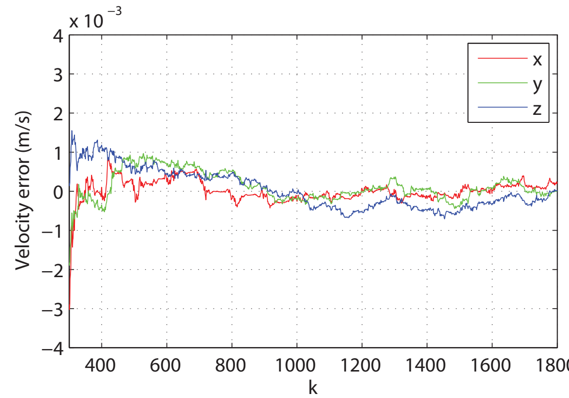


Fig. 6. Steady-state evolution of the ocean current velocity error

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