A Structured Systems Approach for Optimal Actuator-Sensor Placement in Linear Time-Invariant Systems

Sérgio Pequito [1,2] Soummya Kar [1] A. Pedro Aguiar [2,3]

Abstract— **In this paper we address the actuator/sensor allocation problem for linear time invariant (LTI) systems. Given the structure of an autonomous linear dynamical system, the goal is to design the structure of the input matrix (commonly denoted by B) such that the system is structurally controllable with the restriction that each input be dedicated, i.e., it can only control directly a single state variable. We provide a methodology to determine the minimum number of dedicated inputs required to ensure structural controllability, and characterize all (when not unique) possible configurations of the minimal input matrix B. Furthermore, we show that the proposed solution incurs polynomial complexity in the number of state variables. By duality, the solution methodology may be readily extended to the structural design of the corresponding minimal output matrix (commonly denoted by C) that ensures structural observability.**

I. INTRODUCTION

This paper is motivated by the dearth of scalable techniques for the analysis and synthesis of different large-scale complex systems, notably ones which tackle design and decision making in a single framework. Examples include power systems, public or business organizations, large manufacturing systems, wireless control systems, biological complex networks, and formation control, to name a few. Focusing on the last case, consider, for instance, the synchronization problem in vehicular formations: Given a communication topology for inter-vehicle information exchange and the individual vehicle (agent) models, we are often interested in addressing the following questions:

- What is the smallest subset of agents (and specifically which ones), that need a *dedicated input* (i.e., a control that directly affects a single state variable), such that the system is controllable?
- Similarly, what is the smallest subset of agents (and more specifically which ones) that need to be equipped with a *dedicated output* (i.e., an output that measures directly a single state variable), such that the entire network state may be estimated?

Referring to the vehicular-formation scenario, the different agents that require dedicated controls to achieve system controllability, are those that play the role of *leaders*. Under infrastructure and operational constraints, identifying the smallest subset of such agents clearly maximizes the efficiency of the system. The concerns posed above go beyond the vehicular-formation example and are applicable to wider classes of large-scale multi-agent scenarios.

To address these problems, we will resort to structural systems theory [1], in which the main idea is to reformulate and study of an equivalent class of systems for which system-theoretic properties are investigated based on only the sparsity pattern (i.e., the location of zeroes and non-zeroes) of the state space representation matrices. Such an approach is particularly helpful when dealing with systems parameter uncertainties. Analysis using structural systems provides system-theoretic guarantees that hold for almost all values of the parameters, except for a manifold of zero Lebesgue measure [2]. Properties such as controllability and observability are referred to as *structural controllability*[1] and *structural observability* in this framework, as they hold in general, i.e, for almost all non-zero entries in the state space representation. With this, our design objective may be precisely formulated as follows:

Problem Statement

Given

$$\dot{x} = \bar{A}x, \tag{1}$$

where $\bar{A}$ represents the structural pattern of A (i.e., the locations of zeroes and non-zeroes only), our goal is

$\mathcal{P}_1$ Design $\bar{B}$ (i.e., find the structural pattern) with a minimum number of dedicated inputs, such that $(\bar{A}, \bar{B})$ is structurally controllable. Stated formally, characterize (all) $\overline{B}\quad\mathbb{R}^{n\times p}$ such that : given $j\quad\{1,\cdots,p\}$ then

$$ {}^{1}_{i\ \{1,\cdots,n\}}\ \bar{B}_{ij} = 0,$$

or, in other words,

$$\dot{x} = \bar{A}x + {}^{p}_{j=1}\ \bar{b}_{i_j}u_j,\ i_j\quad\{1,\cdots,n\}, \tag{2}$$

This work was partially supported by grant SFRH/BD/33779/2009, from Fundação para a Ciência e a Tecnologia (FCT) and the CMU-Portugal (ICTI) program, and by projects CONAV/FCT-PT (PTDC/EEACRO/113820/2009), FCT (PEst-OE/EEI/LA0009/2011), and MORPH (EU FP7 No. 288704).

[1] Dept. of Electrical and Computer Engineering, Carnegie Mellon University, Pittsburgh, PA 15213.

[2] Institute for System and Robotics, Instituto Superior Técnico, Technical University of Lisbon, Lisbon, Portugal.

[3] Faculty of Engineering, University of Porto (FEUP), Portugal.

[1] A pair (A, B) is said to be structurally controllable if there exists a pair $(A\ , B\ )$ with the same structure as (A, B), i.e., same locations of zeroes and non-zeroes, such that $(A\ , B\ )$ is controllable. By density arguments, it may be shown that if a pair (A, B) is structurally controllable, then almost all (with respect to the Lebesgue measure) pairs with the same structure as (A, B) are controllable. In essence, structural controllability is a property of the structure of the pair (A, B) and not the specific numerical values. A similar definition and characterization holds for structural observability (with obvious modifications).

where $\bar{B} = [b_{i_1} \ \cdots \ b_{i_p}]$, $\bar{b}_{i_j}$ represents the i_j-th canonical vector and $u_j \quad \mathbb{R}$ represents the j-th control, such that system (2) is structurally controllable, and there exists no other $p \ < p$ that satisfies the previous requirement.

Solution of $\mathcal{P}_1$ also addresses the corresponding optimal (minimal placement) structural observability output matrix design problem by invoking the duality between estimation and control in LTI systems.

The literature on structured systems theory is extensive; see [2-5] for earlier work, see also [1] for a recent survey. For applications to optimal sensor and actuator placements, the reader may refer to [6] and references therein; however, these approaches mostly lead to combinatorial implementation complexity in the number of state vertices (agents), or are often based on simplified heuristic-based reductions of the optimal design problems. Systematic approaches to structured systems based design were investigated recently in the context of different application scenarios, see, for example, [7-12]; for instance, in network estimation, as in [7,10], where strategies for output (sensor) placement are provided, ensuring only sufficient (but not necessarily minimal) conditions for structural observability, whereas in [7,11] applications to power system state estimation are explored. From the structural observability viewpoint, as a key contrast to the above approaches, we study the constrained output placement problem, specifically, in which the outputs are dedicated, in that, they may only measure a single state variable. The formulation that is closest to our setup in terms of minimal actuator placement arises from biological complex networks [12], where the concept of a *driving vertex* is introduced, i.e., a state vertex through which a subset of the state variables (seen as state vertices) can be controlled. One of the problems addressed in that work may be stated as follows: What is the minimum number of driving vertices that make the entire network controllable? However, the results in [12] hold only for the case in which the system graph is strongly connected. In contrast, in this paper, in addition to providing the minimum number of dedicated inputs (driving vertices) for generic system matrices (digraphs), we also characterize the set of all possible minimal dedicated feasible configurations, i.e., the minimum subset of state variables that ensures structural controllability by assigning to them dedicated inputs.

To summarize, the main contributions of this paper include: 1) the identification of the minimum number of dedicated inputs (to ensure structural controllability); 2) characterization of all such feasible minimal dedicated input placement configurations (i.e., the design of $\bar{B}$ up to column permutations); 3) algorithmic generation of a minimal feasible dedicated input configuration in polynomial complexity (in the number of state variables).

The rest of the this paper is organized as follows. Section II reviews some concepts and introduces results (some of them new) in structural systems theory and establish their relations to graph-theoretic constructs. Subsequently, in Section III we present the main technical results (the proofs are omitted, they are provided in the extended version [13], see also [14]), followed by an illustrative example in Section IV. Conclusions are presented in Section V.

II. PRELIMINARIES AND TERMINOLOGY

In this section we recall some classical concepts in structural systems, introduced in [3].

Given a dynamical system (1), an efficient approach to the analysis of its structural properties is to associate it with a directed graph (digraph) $\mathcal{D} = (V, E)$, in which V denotes a set of *vertices* and E represents a set of *edges*, such that, an edge (v_j, v_i) is directed from vertex v_j to vertex v_i. Denote by $\mathcal{X} = \{x_1, \cdots, x_n\}$ and $\mathcal{U} = \{u_1, \cdots, u_p\}$ the set of state vertices and input vertices, respectively. Denote by $\mathcal{E}_{\mathcal{X},\mathcal{X}} = \{(x_i, x_j) : \ [\bar{A}]_{ji} = 0\}$ and $\mathcal{E}_{\mathcal{U},\mathcal{X}} = \{(u_j, x_i) : \ [\bar{B}]_{ji} = 0\}$, to define $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ and $\mathcal{D}(\bar{A}, \bar{B}) = (\mathcal{X} \quad \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \quad \mathcal{E}_{\mathcal{U},\mathcal{X}})$. A digraph $\mathcal{D}_s = (V_s, E_s)$ with $V_s \quad V$ and $E_s \quad E$ is called a *subgraph* of $\mathcal{D}$. If $V_s = V$, $\mathcal{D}_s$ is said to *span* $\mathcal{D}$. A sequence of edges $\{(v_1, v_2), (v_2, v_3), \cdots, (v_{k-1}, v_k)\}$, in which all the vertices are distinct, is called *an elementary path* from v_1 to v_k. When v_k coincides with v_1, the sequence is called a *cycle*.

In addition, we will require the following graph theoretic notions [15]: A digraph $\mathcal{D}$ is said to be strongly connected if there exists a directed path between any two pairs of vertices. A strongly connected component (SCC) is a maximal subgraph $\mathcal{D}_S = (V_S, E_S)$ of $\mathcal{D}$ such that for every $v, w \quad V_S$ there exists a path from v to w and from w to v. Note that, an SCC may have several paths between two vertices and the path from v to w may comprise some vertices not in the path from w to v. Visualizing each SCC as a virtual node (or supernode), one may generate a *directed acyclic graph* (DAG), in which each node corresponds to a single SCC and a directed edge exists between two SCCs *iff* there exists a directed edge connecting the corresponding SCCs in the original digraph. The DAG associated with $\mathcal{D} = (V, E)$ may be efficiently generated in $\mathcal{O}(|V| + |E|)$ [15], where $|V|$ and $|E|$ denote the number of vertices in V and the number of edges in E, respectively. The SCCs in a DAG can be characterized as follows

Definition 1: An SCC is said to be linked if it has at least one incoming/outgoing edge from another SCC. In particular, an SCC is *non-top linked* if it has no incoming edges to its vertices from the vertices of another SCC and *non bottom linked* if it has no outgoing edges to another SCC.

For any two vertex sets $S_1, S_2 \quad V$, we define the *bipartite graph* $\mathcal{B}(S_1, S_2, E_{S_1,S_2})$ associated with $D = (V, E)$, to be a directed graph (bipartite), whose vertex set is given by $S_1 \quad S_2$ and the edge set E_{S_1,S_2} by $E_{S_1,S_2} = \{(s_1, s_2) \quad E \ : \ s_1 \quad S_1, s_2 \quad S_2 \ \}$.

Given $\mathcal{B}(S_1, S_2, E_{S_1,S_2})$, a matching M corresponds to a subset of edges in E_{S_1,S_2} that do not share vertices, i.e., given edges $e = (s_1, s_2)$ and $e \ = (s_1, s_2)$ with $s_1, s_1 \quad S_1$ and $s_2, s_2 \quad S_2$, $e, e \quad M$ only if $s_1 = s_1$ and $s_2 = s_2$. A maximum matching M may then be defined as a matching M that has the largest number of edges among all possible matchings. The maximum matching problem may be solved efficiently in $\mathcal{O}(\ \overline{|S_1 \quad S_2|}|E_{S_1,S_2}|)$ [15]. Vertices in S_1 and S_2 are *matched vertices* if they belong to an edge in the maximum matching M , otherwise, we designate the vertices as *unmatched vertices*. If there are no unmatched vertices, we say that we have a *perfect match*. It is to be noted that a maximum matching M may not be unique.

For ease of referencing, in the sequel, the term *right-unmatched vertices* (w.r.t. $\mathcal{B}(S_1, S_2, E_{S_1,S_2})$ and a maximum matching M) will refer to only those vertices in S_2 that do not belong to a matched edge in M .

A. Structural Systems

Given a digraph $\mathcal{D}(\bar{A})$ and $\mathcal{D}(\bar{A}, \bar{B})$ (when appropriate), we further define the following special subgraphs [3]:

- *State Stem* - An elementary path composed exclusively by state vertices, or a single state vertex.

- *Input Stem* - An elementary path composed of an input vertex (the root) linked to the root of a state stem.

- *State Cactus* - Defined recursively as follows: A state stem is a state cactus. A state cactus connected to a cycle from any point other than the tip is also a state cactus.

- *Input Cactus* - Defined recursively as follows: An input stem with at least one state vertex is an input cactus. An input cactus connected to a cycle from any point other than the tip is also an input cactus.

- *Chain* - A group of disjoint cycles (composed by state vertices) connected to each other in a sequence, or a single cycle.

The root and the tip of a stem are also the root and tip of the associated cactus.

Furthermore, recall the following result:

Theorem 1 ([16]): For an LTI system $\dot{x} = Ax + Bu$, the following statements are equivalent:

i) The corresponding structured linear system $(\bar{A}, \bar{B})$ is structurally controllable.
ii) The digraph $\mathcal{D}(\bar{A}, \bar{B})$ is spanned by a disjoint union of input cacti.

Note that, by definition, an input cactus may have an input vertex linked to several state vertices, i.e., the input vertex may connect to the root of a state stem (i.e., input stem) and could be linked to one or more states in a chain.

B. Relation between Maximum Matching and Concepts in Structural Systems

The following results provide a bridge between structural systems concepts and graph constructs such as maximum matching. These results will be used to characterize the minimal dedicated input configurations in Section III.

Lemma 1: Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ and M a maximum matching associated with $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$. Then, if M is a perfect match, the edges in M correspond to a disjoint union of cycles in $\mathcal{D}(\bar{A})$.

Lemma 2 (Maximum Matching Decomposition): Consider the digraph $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ and let M be a maximum matching associated with the bipartite graph $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$. Suppose M consists of a non-empty set of right-unmatched vertices. Then, M (more precisely, the edges in M), together with the set of isolated vertices, constitutes a disjoint union of cycles and state stems (with roots in the right-unmatched vertices and tips in the left-unmatched vertices) that span $\mathcal{D}(\bar{A})$. Moreover, such a decomposition is *minimal*, in the sense that, no other spanning subgraph decomposition of $\mathcal{D}(\bar{A})$ into state stems and cycles contains strictly fewer number of state stems.

In other words, the maximum matching problem leads to two different kinds of matched edge sequences in M ; sequences of edges in M starting in right-unmatched state vertices, and the remaining sequences of edges that start and end in a matched vertices. These sequences represent state stems and cycles, respectively.

In case a graph is composed of multiple SCCs, we define

Definition 2: Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ and M a maximum matching associated with $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$. A non-top linked SCC is said to be a *top assignable SCC* if it contains at least one right-unmatched vertex (with respect to M).

Note that the total number of top assignable SCCs may depend on the particular maximum matching M (not unique in general) under consideration; as such we may define:

Definition 3: Consider the digraph $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$. The *maximum top assignability index* of $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ is the maximum number of top assignable SCCs among the maximum matchings M associated with $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$. Similarly, the *maximum bottom assignability index* of $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ is the maximum number of bottom assignable SCCs among the maximum matchings M associated with $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$.

III. MAIN RESULTS

In this section we present the main results of this paper (due to space limitations, proofs are provided in the extended version [13], see also [14]), broadly centered on the following two issues:

- Determine the minimum number of dedicated inputs to be allocated to ensure structural controllability.
- Describe the set of all possible *minimal feasible dedicated input configurations* (i.e., allocation configurations with the minimum number of dedicated inputs) which lead to structural controllability.

The first result relates the minimum number of dedicated inputs necessary to ensure structural controllability to the structure of the cacti associated with the system digraph.

Theorem 2: Given the system (state) digraph $\mathcal{D}(\bar{A})$, the minimum number of dedicated inputs required to ensure structural controllability is equal to the minimum number of disjoint state cacti that span $\mathcal{D}(\bar{A})$.

Theorem 2 reduces the problem of finding the minimum number of dedicated inputs to that of finding the minimum number of disjoint state cacti spanning $\mathcal{D}(\bar{A})$. The next set of results are concerned with explicitly characterizing the minimal feasible configurations by invoking the relationship (see, for instance, Lemma 2) between cacti decompositions and more readily computable graph constructs such as maximum matching.

Minimum Number of Dedicated Inputs

The following characterization of the minimum number of disjoint state cacti spanning $\mathcal{D}(\bar{A})$ (and, hence, the minimum number of dedicated inputs) holds.

Theorem 3 (Minimum Number of Dedicated Inputs): Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ be the system digraph with non-top linked SCCs in its DAG representation. Let M be a maximum matching associated with the bipartite graph $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ and let $\mathcal{V}$ $\mathcal{X}$ be the set of corresponding right-unmatched vertices. Then, the minimum number of dedicated inputs p is given by

$$p = m + \quad - \quad , \tag{3}$$

where $m = |\mathcal{V}|$ and denotes the maximum top assignability index of $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$.

Note that, in Theorem 3, the number of right-unmatched vertices (and hence, the minimum number of dedicated inputs p) does not depend on the specific instantiation of the maximum matching M being considered (which is not unique in general). Moreover, it may be readily verified from the definitions, that if $\mathcal{D}(\bar{A})$ is strongly connected, we have $= 1$, in Theorem 3, and may only assume two values, 0 or 1, depending on whether $m = 0$ or $m = 1$ respectively. As such, Theorem 3 may be simplified significantly if $\mathcal{D}(\bar{A})$ is known to be strongly connected.

Corollary 1: Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ be strongly connected and let M be a maximum matching associated with $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$. Designate by $\mathcal{V}$ the set of right-unmatched vertices and let $m = |\mathcal{V}|$. Then, the number of dedicated inputs p is

- 1 if $m = 0$ (i.e., there is a perfect match),
- m if $m > 0$.

Characterizations of the required number of minimal dedicated inputs similar to Corollary 1 were presented in [12] (see, Theorem 2 in the supplement material of [12]).

Theorem 3 provides the minimum number of required dedicated inputs, hence the minimum number of columns in $\bar{B}$ (each with only one non-zero entry) required to ensure structural controllability. We now explicitly characterize all such $\bar{B}$'s (up to a permutation of the columns). Each such combination will be referred to as a *minimal feasible dedicated input configuration.*

Minimal Feasible Dedicated Input Configurations

A minimal feasible dedicated input configuration will be denoted by a subset $\mathcal{S}_u = \{x_{i_1}, \cdots, x_{i_p}\}$ of states, where p corresponds to the minimal number of dedicated inputs ensuring structural controllability (see (3) in Theorem 3) and i_k $\{1, \cdots, n\}$ for all $k = 1, \cdots, p$. In other words, a subset $\mathcal{S}_u$ corresponds to a minimal feasible dedicated input configuration if allocating dedicated actuators (inputs) to each of the p states x_{i_k} in the subset leads to structural controllability. Also, note that each such subset corresponds to a unique canonical $\bar{B}$; hence, identifying the set of all possible canonical minimal $\bar{B}$'s is equivalent to identifying all such subsets of minimal feasible dedicated input configurations.

Also, denote by the set of all possible minimal feasible dedicated input configurations, i.e.,

$$\begin{aligned} =\{&\{x_{i_1}, \cdots, x_{i_p}\} : x_{i_1} = \cdots = x_{i_p} \text{ and if a} \\ &\text{dedicated input is assigned to each } x_{i_k}, \text{where} \\ &i_k \quad \{1, \cdots, n\} \text{ and } k = 1, \cdots, p, \text{ the resulting} \\ &\text{LTI system is structurally controllable}\}. \end{aligned}$$

Note that, by the above definition, a minimal feasible dedicated input configuration is invariant to any permutation of the states in its associated configuration representation; a permutation of the states in the configuration leads to the same dedicated input assignment. The following set of results concerns the efficient description and enumeration of the set of all possible minimal feasible dedicated input configurations. The key driving factor behind an efficient representation of is the existence of subsets ^{j} $\mathcal{X}$, $j = 1, \cdots, p$, such that is *almost* (to be made precise soon) the Cartesian product of the ^{j}'s, i.e., ${}^{1} \times \cdots$ ^{p}, i.e., it will be shown that, up to permutation and some *natural constraints* on the ^{j}'s, a subset $\mathcal{S}_u = \{x_{i_1}, \cdots, x_{i_p}\}$ belongs to *iff* x_{i_j} ^{j} for all $j = 1, \cdots, p$. Specifically, we have the following:

Theorem 4 (Naturally Constrained Partitions): Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ be a digraph with $|\mathcal{X}| = n$ and $\mathcal{N}^i = (\mathcal{X}^i, \mathcal{E}_{\mathcal{X}^i,\mathcal{X}^i})$, for $i = 1, \cdots,$, be the non-top linked SCCs of the DAG representation of $\mathcal{D}(\bar{A})$, with $\mathcal{X}^i$ $\mathcal{X}$ and $\mathcal{E}_{\mathcal{X}^i,\mathcal{X}^i}$ $\mathcal{E}_{\mathcal{X},\mathcal{X}}$. In addition, let M be a maximum matching associated with the bipartite graph $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ with $m = |\mathcal{V}|$ right-unmatched vertices, where $\mathcal{V} = \{v_1, v_2, \ldots, v_m\}$ $\mathcal{X}$ is the set of right-unmatched vertices with respect to M and p denotes the minimum number of dedicated inputs as in Theorem 3.

There exist subsets ^{j} $\mathcal{X}$, $j = 1, \cdots, p$, given by

- $j = 1, \cdots, m$:

$$\begin{aligned} {}^{j} =\{x :\ & (\mathcal{V} - \{v_j\}) \quad \{x\} \text{ for } x \quad \mathcal{X} \text{ is the set of} \\ & \text{right-unmatched vertices for some} \\ & \text{maximum matching of } \mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})\}, \end{aligned}$$

- $j = m + 1, \cdots, p :$ $\quad {}^{j} = \underset{l \quad \{1, \cdots, \quad \}}{\qquad} \mathcal{N}^l,$

such that, the set of minimal feasible input configurations may be characterized as follows: The subset $\mathcal{S}_u = \{x_{\ (1)}, \cdots, x_{\ (p)}\}$ $\mathcal{X}$ is a member of if and only if the following *natural constraints* hold:

(i) x_{i_j} ^{j}, for $j = 1, \cdots, p$;

(ii) x_{i_j} ^{j} and $x_{i_{j'}}$ ${}^{j'}$ for $j = j'$ with 1 j, j' m implies that x_{i_j} and $x_{i_{j'}}$ are the root to two different minimal state stems with respect to a possible maximum matching of $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$;

(iii) each non-top linked SCC $\mathcal{N}^l$ has at least one state variable in $\mathcal{S}_u$ that belongs to $\mathcal{N}^l$.

Note that the sets ^{j} are defined on the basis of the specific maximum matching M in consideration. However, as it is evident from the proofs, up to a permutation of the indices j, $j = 1, \cdots, p$, the ^{j}'s are independent of the actual instantiation of M (which may not be unique). We refer to the sets ^{j}, for $j = 1, \cdots, p$ endowed with the *natural constraints* as the *natural constrained partitions* of . Given the description in Theorem 4, we are interested in understanding the computational (algorithmic) complexity of implementing the *natural constrained partitions*, as well as understanding how to use such characterization to compute iteratively a minimal feasible dedicated input configuration. This is the scope of the next result.

Theorem 5 (Complexity): Let the hypotheses of Theorem 4 hold. Then, there exist algorithms of polynomial complexity (in the number of state vertices) to implement the following procedures:

1) obtaining the minimum number of dedicated inputs;
2) constructing the natural constrained partitions, ^{j}'s with the natural constraints;
3) generating a minimal feasible dedicated input configuration, iteratively.

First, note that, although by Theorem 5, there exist polynomial algorithms to constructing the ^{j}'s and obtaining a minimal feasible dedicated input configuration, listing specifically all possible minimal configurations may be combinatorial (the number of such configurations could be exponential in the number of state vertices). The algorithmic procedures to obtaining the minimum number of dedicated inputs and the natural constrained partitions can be obtained in the extended version [13], see also [14]. An example presented in the following section.

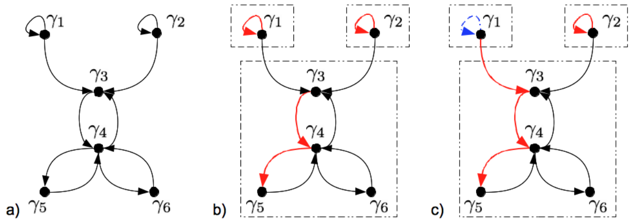

Fig. 1. a) The digraph $\mathsf{D}(\bar{A})$ modeling the inter-dependecy of the parameters of the 6 agents; b) The SCCs are depicted by the rectangles and the red edges represent a maximum matching (recall Lemma 2); c) Removing the edge in blue from $\mathsf{D}(\bar{A})$ forces ${}_1$ to be a right-unmatched vertex in a maximum matching associated with a bipartite graph without the incoming edge in ${}_1$.

IV. AN ILLUSTRATIVE EXAMPLE

The following example illustrates the procedure to obtaining the minimum number of dedicated inputs. Consider a set of 6 agents that do share information. Let us assume that each agent i has a predefined path $y_i : \mathbb{R} \quad \mathbb{R}^N$, to follow parametrized by a time-dependent parameter ${}_i : \mathbb{R} \quad \mathbb{R}$, such that $y_i({}_i(t))$ provides the position of an agent i at time instant t. Suppose we are interested in addressing the controlled synchronization problem, which consists of some predefined parameter specification $\quad \mathbb{R}^6$.

Moreover, suppose that the autonomous system may be described as follows:

$$\begin{aligned} \dot{}_1 &= -k_1 \ {}_1, \\ \dot{}_2 &= -k_2 \ {}_2, \\ \dot{}_5 &= -k_9 \ {}_4, \\ \dot{}_6 &= -k_{10} \ {}_4, \\ \dot{}_3 &= -k_3 \ {}_1 - k_4 \ {}_2 - k_5 \ {}_4, \\ \dot{}_4 &= -k_6 \ {}_3 - k_7 \ {}_5 - k_8 \ {}_6, \end{aligned} \tag{4}$$

where $k_i \quad \mathbb{R}^+$, $i = 1, \cdots, 10$ are prescribed gains.

The question that we address is the following: what is the minimum subset of agents that we need to assign a dedicated control such that any specified is achievable in finite time?

From (4) we obtain the structured (autonomous) system

$$\dot{} = \underbrace{\begin{array}{cccccc} \times & 0 & 0 & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & 0 & 0 \\ \times & \times & 0 & \times & 0 & 0 \\ 0 & 0 & \times & 0 & \times & \times \\ 0 & 0 & 0 & \times & 0 & 0 \\ 0 & 0 & 0 & \times & 0 & 0 \end{array}}_{\bar{A}} \tag{5}$$

where $= [\ {}_1 \ {}_2 \ {}_3 \ {}_4 \ {}_5]^T$ and $\times$ denote the non-zero entries. System (5) provides the digraph representation $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ as indicated in Fig. 1-a).

We consider the following steps:

Step 1 Compute a maximum matching M associated with $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$. Figure 1-b) represents in red, the edges belonging to the maximum matching M (note that M is not unique in general).

Step 2 Find the minimum number of dedicated inputs required to ensure structural controllability. To this end, given M, the set of right-unmatched vertices are $\mathcal{V} = \{\ {}_3, \ {}_6\}$, and, hence, in Theorem 5 we have $m = 2$ and $= 2$ corresponding to the non-top linked SCCs. To find in Theorem 5, consider the "alternatives" for ${}_3$, that are in the non-top linked SCCs. For instance, let us verify if ${}_1$ is a possible alternative to ${}_3$, i.e., if there is another maximum matching with ${}_3$ as one of its right-unmatched vertices and some other state vertex different from ${}_1$, that belong in particular to a non-top linked SCC. To this end, we force ${}_1$ to be a right-unmatched vertex, by removing all incoming edges to ${}_1$ on the original digraph (represented in blue in Fig. 2) and compute a new maximum matching, say M^1. This new maximum matching is depicted in Figure 1-c). Since, the new maximum matching consists of the same number of edges, ${}_1$ is a possible alternative and the SCC containing ${}_1$ is assignable. Now, consider ${}_6$ and let us verify if there exists an alternative in the corresponding non-top linked SCC. Recall that ${}_1$ is fixed because it's in an assignable non-top linked SCC and our goal is to find the maximum assignability index. Let us consider that ${}_2$ is also fixed, (by removing its self loop), and compute a new maximum matching. This provides $\{\ {}_1, \ {}_2, \ {}_3\}$ as right-unmatched vertices, which implies that the maximum matching has less one edge with respect to the maximum matching M^1. Since there is only one non-top linked assignable SCC, in Theorem 5 we have $= 1$. Hence, we need $p = m + \quad - \quad = 2 + 2 - 1 = 3$ dedicated inputs.

Step 3 In this step, we characterize all possible feasible minimal input configurations (up to permutation) by resorting to Theorem 4. Since, $_1$ and $_6$ comprise a possible feasible dedicated input configuration, follows that $_1$ ^{1} and $_6$ ^{2}. In order to extend these sets, i.e., which "alternatives" to $_1$ provide maximum matchings with the same number of edges as M . Consider the original $\mathcal{B}(\mathcal{X},\mathcal{X},\mathcal{E}_{\mathcal{X},\mathcal{X}})$ and fix $_6$ (we are just exploring alternatives to $_1$, as defined in Theorem 4), by removing the edges in $\mathcal{E}_{\mathcal{X},\mathcal{X}}$ that end in $_6$. Moreover, to explore if the remaining vertices $\{\ _2,\ _3,\ _4,\ _5\}$ are viable alternatives, consider for $i = 2,3,4,5$, $\mathcal{B}^{\ _i,\ _6}(\mathcal{X},\mathcal{X},\mathcal{E}_{\mathcal{X},\mathcal{X}} - \{(\cdot,\ _6) - \{(\cdot,\ _i)\}\})$, and $M^{\ _i,\ _6}$ the corresponding (arbitrary) maximum matching. It turns out that $|M^{\ _2,\ _6}| = |M^{\ _3,\ _6}| = |M^{\ _5,\ _6}| = 4$ and $|M^{\ _3,\ _6}| = 3$, hence $^1 = \{\ _1,\ _2,\ _3,\ _5\}$. Similarly, by fixing $_1$ we have $^2 = \{\ _6,\ _5\}$. Because $\quad - \quad = 1$ in Theorem 3, by Theorem 4 we need an extra partition given by $^3 = \{\ _1,\ _2\}$. Together with the natural constraints we have the characterization of all minimal feasible dedicated input configurations.

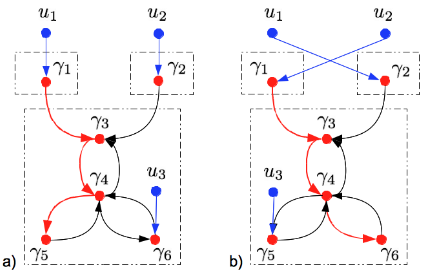

Fig. 2. Digraph $\mathsf{D}(\bar{A})$ where the SCCs are depicted by rectangles. The red edges and state vertices identify the state stems and the blue vertices correspond to the dedicated inputs connected to the roots of the state stems. In a) we have the feasible minimum input configuration $\mathsf{S}_{\mathrm{u}}^2 = \{\ _1,\ _2,\ _6\}$ and in b) the feasible minimum input configuration $\mathsf{S}_{\mathrm{u}}^1 = \{\ _2,\ _1,\ _5\}$.

Step 4 In this step we create iteratively the feasible configurations from the natural constrained partitions in Step 3). The natural constraints impose that first we assign a dedicated input to at least one state variable to non-top linked assignable SCC. Thus,

(i) Picking $_1$ from ^{1}, followed by $_2$ from ^{3} leaves us with the choice of $_5,\ _6$ from ^{2}, leading to the minimal dedicated feasible configurations $\mathcal{S}_u^1 = \{\ _1,\ _2,\ _5\}$ and $\mathcal{S}_u^2 = \{\ _1,\ _2,\ _6\}$, respectively. Equivalently, the above correspond to following structures of the matrix $\bar{B}$ (up to column permutations):

$$\bar{B} = \begin{array}{c} \mathcal{S}_u^1=\{\ _1,\ _2,\ _5\} \\ \begin{bmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \\ 0 & 0 & 0 \end{bmatrix} \end{array}, \qquad \bar{B} = \begin{array}{c} \mathcal{S}_u^2=\{\ _1,\ _2,\ _6\} \\ \begin{bmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{bmatrix} \end{array}.$$

(ii) Similarly, picking $_1$ from ^{3}, followed by $_2$ from ^{1} leaves us with the choice of $_5,\ _6$ from ^{2}, hence providing the same configurations as in (i).

V. CONCLUSIONS AND FURTHER RESEARCH

In this paper we provided a systematic method with polynomial complexity (in the number of the state variables) to obtain the minimum number of dedicated inputs (through the structural design of B up to column permutation), and characterize all possible solutions that ensures structural controllability of a given LTI system. By duality, the results extend to the corresponding structural observability output design. A natural extension of the current framework consists of obtaining minimal allocations for general cost constrained placement problems, in which actuator-sensor placements may incur different costs at different state vertices. This problem is more challenging; a natural way to proceed is to modify the constructions of the natural constrained partitions suitably so as to incorporate the non-homogeneous assignment costs.

REFERENCES

[1] J.-M. Dion, C. Commault, and J. V. der Woude, "Generic properties and control of linear structured systems: a survey." *Automatica*, pp. 1125–1144, 2003.

[2] K. J. Reinschke, *Multivariable control : a graph theoretic approach*, ser. Lect. Notes in Control and Information Sciences. Springer-Verlag, 1988, vol. 108.

[3] C. Lin, "Structural controllability," *IEEE Transactions on Automatic Control*, no. 3, pp. 201–208, 1974.

[4] D. D. Siljak, *Large-Scale Dynamic Systems: Stability and Structure*. Dover Publications, 2007.

[5] K. Murota, *Matrices and Matroids for Systems Analysis*, 1st ed. Springer Publishing Company, Incorporated, 2009.

[6] P. Sharon L. and K. Rex K., "Optimization strategies for sensor and actuator placement," National Aeronautics and Space Administration Langley Research Center, Langley, Virginia 23681, Tech. Rep., 1999.

[7] U. A. Khan and A. Jadbabaie, "Coordinated networked estimation strategies using structured systems theory," *Proc. of the 50th IEEE Conference on Decision and Control*, pp. 2112–2117, 2011.

[8] U. A. Khan and M. Doostmohammadian, "A sensor placement and network design paradigm for future smart grids," *4th IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, 2011.

[9] G. J. P. M. Pajic, S. Sundaram and R. Mangharam, "Topological conditions for wireless control networks," *Proc. of the 50th IEEE Conference on Decision and Control*, pp. 2353–2360, 2011.

[10] T. Boukhobza and F. Hamelin, "Observability analysis and sensor location study for structured linear systems in descriptor form with unknown inputs," *Automatica*, vol. 47, no. 12, pp. 2678–2683, 2011.

[11] F. Pasqualetti, A. Bicchi, and F. Bullo, "A graph theoretic characterization of power network vulnerabilities," in *2011 American Control Conference*, San Francisco, CA, USA, June 29 - July 1 2011, pp. 3918 – 3923.

[12] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, "Controllability of complex networks," *Nature*, vol. 473, no. 7346, pp. 167–173, May 2011. [Online]. Available: http://dx.doi.org/10.1038/nature10011

[13] S. Pequito, S. Kar, and A. Aguiar, "A structured systems approach for optimal actuator-sensor placement in linear time-invariant systems," 2013. [Online]. Available: http://arxiv.org/abs/1210.6724

[14] ——, "A framework for structural input/output and control configuration selection of large-scale systems," *Submitted to IEEE Transactions on Automatic Control*, 2013.

[15] T. H. Cormen, C. Stein, R. L. Rivest, and C. E. Leiserson, *Introduction to Algorithms*, 2nd ed. McGraw-Hill Higher Education, 2001.

[16] R. W. Shields and J. B. Pearson, "Structural controllability of multi-input linear systems," *IEEE Trans. Autom. Control*, vol. AC-21, no. 3, 1976.