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Global attitude and gyro bias estimation based on set-valued observers

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ABSTRACT

The problem of attitude and rate gyro bias estimation is addressed by resorting to measurements acquired from rate gyros and vector observations. A Set-Valued Observer (SVO) is proposed that has no singularities and that, for any initial conditions, provides a bounding set with guarantees of containing the actual (unknown) rotation matrix. The sensor readings are assumed to be corrupted by bounded measurement noise and constant gyro bias. Conditions for the boundedness of the estimated sets are established and implementation details are discussed. The feasibility of the technique is demonstrated in simulation.

1. Introduction

Attitude estimation is an essential element in many modern platforms such as aircrafts, satellites, unmanned air vehicles, and underwater autonomous robots. Usually, solutions for this problem require either noise-free sensor measurements or a stochastic description of the exogenous disturbances and measurement noise. However, if this information is not available *a priori* and only norm-bounds on the disturbances are known, it is desirable to compute explicit bounds on the attitude of the vehicle. Such bounds are suitable, for instance, to be used in robust control designs techniques, where *worst-case* guarantees are provided regarding the performance of the closed-loop system—see, for instance, [1].

There exists a wide variety of attitude estimation techniques in the literature [2]. While some of them, like nonlinear observers, have a deterministic nature [3–7], others take advantage of the stochastic description of the exogenous disturbances and measurement noise to provide (sub-)optimal estimates of the attitude [8–10]. In the latter case, the uncertainty in the measurements is typically modeled as additive Gaussian noise. However, the stochastic characterization of the sensor noise and system disturbances may not be available in some cases, while magnitude bounds are typically known. In these circumstances, optimal stochastic state estimation is not achievable, and, thus, the objective of the estimator is rather to obtain a set of possible state values, given the sensor information. The work in [11] discusses the state estimation problem for systems with bounded inputs, while in [12,13] a similar problem, but using a set-membership description for model uncertainty, is addressed. Recent advances in the framework of these estimators, referred to as Set-Valued Observers (SVOs) [14], are presented in [15–18]. The work in [19] exploits a different approach and proposes an attitude estimator where uncertainty ellipsoids bound the sensor measurements and the filter state. This observer has the advantage of considering also the rigid body dynamics in the filter, which may render it more accurate. However, it has the disadvantage of constraining the moments acting on the rigid body to be generated by an attitude dependent potential, while relying on the linearization of the system to propagate the uncertainty ellipsoids.

The main contribution of the work presented in this paper is the development of an attitude estimator based on SVOs, which relies on vector observations and rate gyros measurements and where these measurements are assumed to be corrupted by bounded noise. A solution is proposed that considers uncertainties defined by polytopes and that guarantees the true state of the system inside the estimated set, as long as the assumptions on the bounds on

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the measurements are satisfied. No linearization is required and discretization errors are directly taken into account in the SVO framework. Preliminary versions of these results were presented in [20,21].

The remainder of this article is organized as follows. In Section 2, the attitude estimation problem is introduced, the available sensor information is described, and a discretization of the system dynamics is proposed. The SVO for attitude estimation with noisy angular velocity measurements is derived in Section 3. Several implementation issues are discussed in Section 4. In Section 5, the performance of the proposed solution is illustrated in simulation for a typical trajectory. Finally, Section 6 presents some concluding remarks.

2. Problem formulation

In this section, we introduce the attitude estimation problem using vector observations and biased angular velocity measurements. The vector observations provide instantaneous information about attitude, while the angular velocity characterizes its time-evolution.

The objective of the present work is to estimate the smallest set that contains the attitude of a rigid body and the bias of the rate gyros, by using the available sensor suite, i.e., to obtain the set-valued attitude and bias estimates with the smallest possible uncertainty.

2.1. System description

The rotation matrix is adopted for attitude representation as it does not suffer from the problems associated with the many other alternatives, namely, unwinding phenomena and singularities [22,23]. The right-invariant description on the special orthogonal group $SO(3)$ is considered so that the rotation matrix kinematics is given by

$$\dot{\mathcal{R}} = (\omega)^\times \mathcal{R}, \tag{1}$$

where $\mathcal{R} \in SO(3)$ denotes the rotation matrix from the reference frame $\{I\}$ – which is assumed to be inertial – to the body-fixed reference frame $\{B\}$, $\omega \in \mathbb{R}^3$ denotes the angular velocity of $\{I\}$ with respect to $\{B\}$, measured in $\{B\}$, and $(\cdot)^\times$ denotes the skew-symmetric operator, which maps $\mathbb{R}^3$ into $\mathfrak{so}(3)$ – the Lie algebra associated with $SO(3)$ – and satisfies $(v)^\times w = v \times w$, $\{v, w\} \in \mathbb{R}^3$. Note that, to ease the notation, the definition of angular velocity is symmetrical to the one usually adopted in the literature on inertial systems (see, for instance, [2]). A triad of rate gyros, installed in reference frame $\{B\}$, measures $\omega_r(k) \in \mathbb{R}^3$, which denotes the angular velocity corrupted by the bias term $b \in \mathbb{R}^3$ and bounded noise $n(k) \in \mathbb{R}^3$ so that

$$\omega_r(k) = \omega(k) + b + n(k), \quad \|n(k)\|_\infty \le \bar{n} \tag{2}$$

where $\|n(k)\|_\infty$ denotes the maximum absolute value of each element of $n(k)$. The unknown rate gyro bias is modeled by a constant vector, i.e.,

$$\dot{b} = 0. \tag{3}$$

2.2. Discretization of the system

The continuous-time model described by (1) is not suitable to be implemented in a digital system. Consequently, we are rather interested in computing $\mathcal{R}$ and ω at discrete time instants kT, $k \in \mathbb{N}^+$, where $T > 0$ denotes the sampling period. For the sake of clarity, the time dependence of the variables will be simply denoted by k, rather than kT.

The solution of the differential equation (1) at $k + 1$ is given by

$$\mathcal{R}(k + 1) = \Phi(k + 1, k)\mathcal{R}(k), \tag{4}$$

where $\Phi(k + 1, k)$ is the state transition matrix [24] from kT to $(k + 1)T$. The state transition matrix can be computed by the Peano–Baker series [24], although it requires the sum of an infinite series and the knowledge of the angular velocity between sampling times. Thus, an alternative approach is herein pursued. For the discretization errors to remain bounded, the time-rate-of-change of the angular velocity must be bounded.

Assumption 1. The magnitude of the angular acceleration is bounded by a known (but possibly conservative) positive scalar $\bar{\alpha}$, i.e., $\|\dot{\omega}\|_\infty \le \bar{\alpha}$, $\bar{\alpha} \in \mathbb{R}^+$.

In the following lemma, the discretization of $\Phi(k + 1, k)$ and the associated errors are characterized.

Lemma 1. *Under Assumption 1, the state transition matrix $\Phi(k + 1, k)$ satisfies*

$$\Phi(k + 1, k) = \Phi_0(k + 1, k) + \Phi_\Delta(k + 1, k)$$

where $\Phi_0(k + 1, k) = e^{T(\omega(k))^\times}$, $\|\Phi_\Delta(k + 1, k)\|_{\max} \le e^{2T^2\bar{\alpha}/2} - 1$, and where $\|.\|_{\max}$ denotes the maximum norm of matrices.

Proof. See Appendix A.

The function $e^{(\cdot)}$ stands for the (scalar) exponential function, as well as, the exponential map of matrices, which in the $SO(3)$ manifold can be obtained in closed form from the *Rodrigues' formula* [22].

Since the bias is assumed to be constant, the discretization of (3) is simply given by

$$b(k + 1) = b(k).$$

2.3. Measurements

On-board sensors such as magnetometers, star trackers, among others, provide vector measurements expressed in body frame coordinates, i.e., ${}^Bv_i = \mathcal{R}\,{}^Iv_i$, where $i = \{1, \ldots, N_v\}$ and N_v is the number of vector measurements, or in the matrix form,

$${}^BV = \mathcal{R}\,{}^IV, \tag{5}$$

where ${}^BV = [{}^Bv_1 \cdots {}^Bv_{N_v}]$ and ${}^IV = [{}^Iv_1 \cdots {}^Iv_{N_v}]$. The time-dependence of the variables was omitted for the sake of simplicity. Notice that if the linear acceleration can be considered negligible when compared to gravity, the accelerometers' measurements can also be suitable to be used as vector observations. It is assumed that the measurements are corrupted by noise contained inside compact polytopes. Thus, the measurements ${}^Bv_i \in \mathbb{R}^3$, $i = \{1, \ldots, N_v\}$ belong to the convex polytope defined by some real matrix $\tilde{M}_{v_i} \in \mathbb{R}^{m\times 3}$ and some vector $\tilde{m}_{v_i} \in \mathbb{R}^m$, i.e., ${}^Bv_i \in \text{Set}(\tilde{M}_{v_i}, \tilde{m}_{v_i})$, where $\text{Set}(M, m) := \{\alpha \in \mathbb{R}^n : M\alpha \le m\}$. The measurements are thus provided by means of a set, rather than a singleton.

A set containing the value of the rate gyro bias, b, is obtained from the sensors by resorting to (2). Vector $\omega_r(k)$ is given by the rate gyros, the noise is known to be upper bounded by $\bar{n}$, and a set containing $\omega(k)$ is obtained by inverting the attitude kinematics (1)

$$\omega(k) = \text{usk}(\text{logm}(\mathcal{R}(k + 1)\mathcal{R}^T(k))),$$

where $\text{logm}(\cdot) : SO(3) \to \mathfrak{so}(3)$ is the inverse of the exponential map in the special orthogonal group and $\text{usk}(\cdot)$ denotes the unskew operator such that for any $a \in \mathbb{R}^3$ one has $\text{usk}((a)^\times) = a$. The set containing the rate gyro bias, i.e., $b \in \text{Set}(\tilde{M}_b(k), \tilde{m}_b(k))$, can be finally obtained from (2) by resorting to the so-called Fourier–Motzkin projection operator [25] or Minkowski addition [26].

Remark 1. The logarithmic map $\text{logm}(\cdot)$ can be computed by inverting *Rodrigues' formula* [22]

$$\text{logm}(R) = \begin{cases} 0 & \text{if } \theta = 0 \\ \dfrac{\theta}{2\sin(\theta)}(R - R^T) & \text{if } \theta \in (0, \pi), \end{cases} \tag{6}$$

where $R \in SO(3)$ and $\theta = \arccos((\text{tr}(R) - 1)/2) \in [0\,\pi)$. If $\theta = \pi$, the result is not unique. However, as long as the trace of R is different from -1, this formula can be used to obtain upper and lower bounds on $\text{logm}(R)$. A set containing the rotation matrix $\mathcal{R}(k)$ is obtained directly from the state as $\text{vec}(\mathcal{R}(k)) = x_{\mathcal{R}}$, while a set containing $\mathcal{R}(k+1)$ is obtained from the vector measurements, $\mathcal{R}(k+1) = {}^B V(k+1)({}^I V(k+1))^{\dagger}$, where $(\cdot)^{\dagger}$ denotes the pseudo-inverse operator, ${}^I V(k+1)$ is known and $\text{vec}({}^B V(k+1)) \in \text{Set}(\tilde{M}_v(k+1), \tilde{m}_v(k+1))$. By resorting to the Minkowski difference [26], a set containing $R = \mathcal{R}(k+1)\mathcal{R}(k)^T$ is obtained. Then, using (6) and interval analysis computations, one can obtain a bounding set for $\omega(k)$. As the quotient $\frac{\theta}{\sin(\theta)}$ is a monotonically increasing function in the domain $\theta \in [0, \pi)$, the corresponding bounds can be easily computed by evaluating its value at the extremes of the interval of θ. The vector $\eta = \text{usk}(\text{logm}(R))$ represents the so-called exponential coordinates of the rotation matrix R.

Let us define the state vector and the measurement vector, respectively, as

$$x = [x_{\mathcal{R}}^T \; b^T]^T, \qquad y = [y_v^T \; b^T]^T,$$

where $x_{\mathcal{R}} = \text{vec}(\mathcal{R})$, $y_v = \text{vec}({}^B V)$ and where the operator $\text{vec}(Z)$ stacks the columns of the $m \times n$ matrix Z into a $mn \times 1$ vector. The complete measurement model is then given by

$$y(k) = \mathcal{C}(k)x(k), \quad \mathcal{C}(k) = \text{diag}(({}^I V(k) \otimes I_3)^T, I_3),$$

where $\otimes$ denotes the Kronecker product of matrices [1, p. 25]. The measurement vector satisfies $y_v \in \text{Set}(\tilde{M}_v, \tilde{m}_v)$, where $\tilde{m}_v = [\tilde{m}_{v_1}^T \; \cdots \; \tilde{m}_{v_{N_b}}^T]^T$ and $\tilde{M}_v = \text{diag}(\tilde{M}_{v_1}, \ldots, \tilde{M}_{v_{N_b}})$, which denotes the block matrix whose diagonal blocks are $\tilde{M}_{v_1}, \ldots, \tilde{M}_{v_{N_b}}$ and the off-diagonal blocks are zero matrices.

Definition 1. Given ${}^I V$, x is said to be *compatible with the set of observations*, $S = \text{Set}([\tilde{M}_v^T \; \tilde{M}_b^T]^T, [\tilde{m}_v^T \; \tilde{m}_b^T]^T)$, if there exists $y \in S$ such that (2)–(5) are satisfied.

Remark 2. Definition 1 states that a given rotation matrix, $\mathcal{R}$, and rate gyro bias, b, are compatible with the uncertain measurements, if the set of vector measurements contain the inertial vector observations, ${}^I V$, rotated by $\mathcal{R}$, and the angular velocity measurements contain the angular velocity plus the vector b.

In the next lemma, it is shown how the output of the system relates to the state, i.e., the time-varying rotation matrix.

Lemma 2. *Assume that the measurement vector $y(k)$, at each time k satisfies $y(k) \in \text{Set}([\tilde{M}_v^T(k) \; \tilde{M}_b^T(k)]^T, [\tilde{m}_v^T(k) \; \tilde{m}_b^T(k)]^T)$. Then, $\mathcal{R}(k)$ and b are compatible with the measurements if and only if $x(k) \in \text{Set}(\tilde{M}(k), \tilde{m}(k))$, where $\tilde{M}(k) = [(\tilde{M}_v(k)({}^I V \otimes I_3)^T)^T \; \tilde{M}_b^T(k)]^T$ and $\tilde{m} = [\tilde{m}_v^T(k) \; \tilde{m}_b^T(k)]^T$.*

Proof. By resorting to the property $\text{vec}(AXB) = \text{vec}(B^T \otimes A)\text{vec}(X)$, it is concluded that

$$y_v = ({}^I V \otimes I_3)^T x_{\mathcal{R}}. \tag{7}$$

Then, we have that $\tilde{M}_v(k)y_v(k) \leq \tilde{m}_v(k)$ is equivalent to $\tilde{M}_v(k)$ $({}^I V \otimes I_3)^T x_{\mathcal{R}} \leq \tilde{m}_v(k)$. Thus, $\tilde{M}(k)x(k) \leq \tilde{m}(k)$. □

3. Attitude and rate gyro bias estimation using SVOs

In this section, a methodology is proposed for the attitude estimation problem with bounded sensor noise and biased angular velocity measurements.

At each time k, the proposed estimator provides a set containing the current state of the system described by (1)–(3). Our goal is to minimize the volume of the set containing the rotation matrix and the rate gyro bias.

Let $X(k)$ be a polytope containing x at time k, and denote by $B(k)$ the polytope obtained by projecting $X(k)$ into the rate gyro bias coordinates. Define $b_0(k)$ as the geometric center of B and $\delta_b(k)$ as a measure of the uncertainty on the bias so that $\delta_b(k) = b - b_0(k)$. The geometric center can be obtained by solving a linear programming (LP) problem (see [27, Section 8.5.1]). The so-called Chebyshev center of a convex polyhedron is the furthest point from the limits of the polytope.

The system dynamics in (4) can be rewritten as

$$x_{\mathcal{R}}(k+1) = A(k)x_{\mathcal{R}}(k), \tag{8}$$

where $A(k) = I_3 \otimes \Phi(k+1, k)$. Due to the sensor uncertainty, $A(k)$ cannot be determined exactly. However, it can be decomposed into the sum of a known component, $A_0(k)$, and unknown components, $A_i(k)$, $i = 1, \ldots, 9$, which encode the uncertainty in each element of $A(k)$. Thus, $A(k)$ satisfies

$$A(k) = A_0(k) + \sum_{i=1}^{9} A_i(k)\Delta_i(k),$$

for some $|\Delta_i(k)| \leq 1$, where $A_0(k) = I_3 \otimes e^{T(\omega_r(k) - b_0(k))^{\times}}$, $A_i(k) = I_3 \otimes \epsilon E(m, n)$, and $E(m, n)$ denotes the 3×3 matrices whose elements are zeros except for the element m, n which is unitary, $m, n = \{1, 2, 3\}$, $i = 3(n-1) + m$, and $\epsilon = \frac{1}{2}\left(e^{2T\epsilon_1} - e^{2T\epsilon_2}\right) + e^{\frac{T^2\bar{\alpha}}{2}} - 1$, $\epsilon_1 = \bar{\omega}_0 + \bar{b_\delta} + \bar{n}$, $\epsilon_2 = \bar{\omega}_0$, where $\bar{\omega}_0 = \|\omega_r - b_0\|_\infty$, $\bar{b_\delta} = \|b_\delta\|_\infty$. The value of ϵ is obtained by exploiting the results in Lemma 1 and Appendix B.

The full state dynamics is then given by

$$x(k+1) = \mathcal{A}(k)x(k),$$

where for some $|\Delta_i(k)| \leq 1$, $\mathcal{A}(k)$ satisfies

$$\mathcal{A}(k) = \mathcal{A}_0(k) + \sum_{i=1}^{9} \mathcal{A}_i(k)\Delta_i(k), \tag{9}$$

with $\mathcal{A}_0(k) = \text{diag}(A_0(k), I_3)$, $\mathcal{A}_i(k) = \text{diag}(A_i(k), 0_3)$.

Let us define $\Psi_0(k+n, k) = \mathcal{A}_0(k+n-1)\mathcal{A}_0(k+n-2)\cdots\mathcal{A}_0(k)$, and

$$W(k, k+N) = \begin{bmatrix} \mathcal{C}(k) \\ \mathcal{C}(k+1)\mathcal{A}_0(k) \\ \mathcal{C}(k+2)\Psi_0(k+2, k) \\ \vdots \\ \mathcal{C}(k+N)\Psi_0(k+N, k) \end{bmatrix}.$$

To ensure the existence of a non-divergent observer the following observability-like assumption is required—see [15, Propositions 3.1 and 3.2].

Assumption 2. Matrices $\mathcal{A}(k)$ and $\mathcal{C}(k)$ are uniformly bounded and there exist constants $\mu_1, \mu_2 > 0$ and $N \in \mathbb{N}$ such that for all k

$$\mu_1 I \leq W^T(k, k+N-1)W(k, k+N-1) \leq \mu_2 I. \tag{10}$$

Note that this assumption is always satisfied if $\text{span}\{{}^I v_1, \ldots, {}^I v_{N_v}\} = \mathbb{R}^3$.

Let the state $x(k)$ be contained inside the compact convex polytope defined by the known matrix $M(k)$ and vector $m(k)$, i.e., $x(k) \in X(k)$, where $X(k) = \text{Set}(M(k), m(k))$. Due to the presence of noise in the angular velocity measurements, which is reflected in the uncertainty Δ_i, $i = \{1, \ldots, 9\}$, the set of feasible states at time $k+1$ is in general non-convex. Nevertheless, we will see next that, by considering specific realizations of (9) and by using an SVO to obtain the polytope that contains the state for each particular realization, a set containing the state $x(k+1)$ can be derived. As such, consider now a realization of (9) where $\Delta_i(k) = \Delta_i^*, |\Delta_i^*| \le 1, i = \{1, \ldots, 9\}$ and denote by $\mathcal{A}_{\Delta^*}$ the corresponding uncertainty matrix, i.e., $\mathcal{A}_{\Delta^*} = \mathcal{A}_1^*\Delta_1^* + \cdots + \mathcal{A}_9^*\Delta_9^*$. For each $\mathcal{A}_{\Delta^*}$, the technique described in [16] can be used to design an SVO that provides a set-valued estimate of the state of the system. Indeed, if matrix $\mathcal{A}_0(k)+\mathcal{A}_{\Delta^*}$ is non-singular, one can write the following inequality as a constraint on the state $x(k+1)$:

$$\underbrace{M(k)(\mathcal{A}_0(k)+\mathcal{A}_{\Delta^*})^{-1}}_{M_*(k+1)} x(k+1) \le \underbrace{m(k)}_{m_*(k+1)} . \tag{11}$$

In other words, for $\Delta_i(k) = \Delta_i^*, i = \{1, \ldots, 9\}, x(k+1) \in \text{Set}\,(M_*(k+1), m_*(k+1))$.

Let $\nu_i, i = \{1, \ldots, 2^9\}$ denote a vertex of the hyper-cube $\mathcal{H} := \{\delta \in \mathbb{R}^9 : |\delta| \le 1\}$, where $\nu_i = \nu_j \Leftrightarrow i = j$. Moreover, denote by $\hat{X}_{\nu_i}(k+1)$ the set of points $x(k+1)$ that satisfies (11) where $\mathcal{A}_{\Delta^*} = \mathcal{A}_{\nu_i}$ and with $x(k) \in \text{Set}\,(M(k), m(k))$. Notice that $\hat{X}_{\nu_i}(k+1)$ can be obtained from (11). Further define

$$\hat{X}(k+1) := \text{co}\left\{\hat{X}_{\nu_1}(k+1), \ldots, \hat{X}_{\nu_{2^9}}(k+1)\right\}, \tag{12}$$

where $\text{co}\{p_1, \ldots, p_m\}$ is the smallest convex set containing the points $p_1, \ldots, p_m$, also known as the convex hull of $p_1, \ldots, p_m$. Since the set of all possible states at time $k+1$ is, in general, non-convex, we are going to use $\hat{X}(k+1)$ to overbound it. As $\hat{X}(k+1)$ is the convex hull of a finite number of polytopes, it can be written in the form $\text{Set}(\hat{M}(k+1), \hat{m}(k+1))$. By intersecting this set with the set of points compatible with the measurements, it can be concluded that $x(k) \in X(k+1)$, where $X(k+1) = \text{Set}(M(k+1), m(k+1))$, and $M(k+1) = [\hat{M}^T(k+1)\ \tilde{M}^T(k+1)]^T$ and $m(k+1) = [\hat{m}^T(k+1)\ \tilde{m}^T(k+1)]^T$. The following theorem describes the proposed attitude SVO.

Theorem 1. *Assume that there exist matrix $M(k)$ and vector $m(k)$, such that*

$$x(k) \in \text{Set}(M(k), m(k)) \cap \text{vec}(SO(3)) \times \mathbb{R}^3,$$

where $\text{Set}(M(k), m(k))$ is a compact set. Then, under Assumptions 1 and 2, the set $\text{Set}(M(k+1), m(k+1)) \cap \text{vec}(SO(3)) \times \mathbb{R}^3$, as defined previously, is compact and contains all the states $x(k+1)$ (i.e., $\mathcal{R}(k+1)$ and b) that satisfy (1)–(3) and that are compatible with the observations at time $k+1$.

Proof. Define $\check{X}(k+1)$ as the set of all possible states of the system at time $k+1$. Under Assumption 1, one can compute the matrices $\mathcal{A}_i(k)$ in (9). In addition, (11) defines the set of states at time $k+1$ that satisfy (9) and are compatible with the measurements. By evaluating (11) for $\Delta = [\Delta_1 \cdots \Delta_9]$ in the vertices of $\mathcal{H}$, one obtains $\hat{X}_{\nu_1}(k+1), \ldots, \hat{X}_{\nu_{2^9}}(k+1)$. Uniform boundedness of the sets $\hat{X}_{\nu_1}(k+1), \ldots, \hat{X}_{\nu_{2^9}}(k+1)$ is ensured by the observability-like condition in Assumption 2. It was shown in [18] that, since $\Delta(k)$ can be obtained by convex combinations of the vertices of $\mathcal{H}$, the state of the system, $x(k+1)$, is inside the set generated by the convex hull in (12), which is compact since it is the convex hull of compact sets. Thus, $X(k+1)$ is an overbound of $\check{X}(k+1)$ and an overbound of the space of possible solutions of $\mathcal{R}(k+1)$ and b and is given by $\text{Set}(M(k+1), m(k+1)) \cap \text{vec}(SO(3)) \times \mathbb{R}^3$. □

Remark 3. If $\mathcal{A}(k) + \mathcal{A}_{\Delta^*}(k)$ is singular or ill-conditioned, one can write the inequality

$$\underbrace{\begin{bmatrix} I & -\mathcal{A}_0(k) - \mathcal{A}_{\Delta^*}(k) \\ -I & \mathcal{A}_0(k) + \mathcal{A}_{\Delta^*}(k) \\ 0 & M(k) \end{bmatrix}}_{\Lambda} \begin{bmatrix} x(k+1) \\ x(k) \end{bmatrix} \le \underbrace{\begin{bmatrix} 0 \\ 0 \\ m(k) \end{bmatrix}}_{\lambda},$$

and then use the Fourier–Motzkin projection operator [25] to compute $M_*(k+1)$ and $m_*(k+1)$ such that $M_*(k+1)x(k+1) \le m_*(k+1)$, i.e.,

$$(M_*(k+1), m_*(k+1)) = \text{FM}(\Lambda, \lambda, 12),$$

where for any $x \in \mathbb{R}^{n_x}$, $x \in \text{Set}(A, b)$, $(\bar{A}, \bar{b}) := \text{FM}(A, b, n)$ stands for the Fourier–Motzkin projection operator, where $n = n_x - \bar{n}_x > 0$, and $\bar{A}$ and $\bar{b}$ satisfy, for all $\bar{x} \in \mathbb{R}^{\bar{n}_x}$, $\bar{A}\bar{x} \le \bar{b} \Leftrightarrow \exists_{\tilde{x} \in \mathbb{R}^n} : A[\bar{x}^T\ \tilde{x}^T]^T \le b$.

Remark 4. As $\mathcal{C}(k)$ fulfills (10), the estimate of the SVO proposed in this section converges to a bounded set and the observer is global as it converges for any initial conditions.

Remark 5. Assumption 2 clearly holds if $\mathcal{C}(k)$ is invertible, that is, if $\text{span}\{{}^I v_1, \ldots, {}^I v_{N_v}\} = \mathbb{R}^3$. However, the attitude system (7)–(8) is also observable when only a single time-varying vector observation is available if the system is sufficiently excited, i.e., if there is $N > 0$ such that, for any k, $\text{span}\{{}^I v_1(k), \ldots, {}^I v_1(k+N)\} = \mathbb{R}^3$.

4. Implementation issues

In this section, several details on the implementation of the proposed solution are discussed, namely, memory consumption, computational complexity, and strategies to increase performance.

In the proposed methodology, the sizes of $M(k)$ and $m(k)$ may be increasing with time, which can be problematic from the implementation point-of-view. To reduce the number of rows of $M(k)$ and the length of $m(k)$, one should eliminate the linearly dependent constraints. Algorithms to identify redundant constraints can be found in [28,29] and references therein. Even if the linearly dependent constraints are removed, the number of constraints may, theoretically, grow linearly with the number of iterations. The number of constraints can be limited by enclosing the estimated polytope within a polyhedron with a fixed number of vertices. On the other hand, we need not to perform this operation in every iteration. One can decide to carry out the enclosing only every fixed number of iterations, or when the number of linearly independent constraints exceeds a pre-specified limit.

The computational complexity and the accuracy of the estimates depend on the algorithms that perform the operations over the set-valued estimates, namely, the LP problems, the polyhedron enclosure, and the convex hull. The selection of algorithms depends on the available computational resources.

To increase the convergence rate, the uncertainty set of the rate gyro bias can be split, and each sub-set can be used to initialize a different SVO. Since only one SVO contains the true value of the bias, all other SVOs may degenerate into empty sets for the estimates, as time goes by. At that point, the uncertainty space of the bias is again divided. This approach has the additional advantage of being easily implemented in parallel computer systems.

5. Simulation results

In this section, we present simulation results illustrating the performance of the proposed solution. The results of the SVO are compared with the multiplicative extended Kalman filter (MEKF) [8]. Using the strategy proposed in Section 4, the uncertainty set of the rate gyro bias was split into 27 sub-sets.

The trajectory generated is characterized by an angular velocity with the following oscillatory profile:

$$\omega(t) = \begin{bmatrix} 4.01\sin(2\pi 0.05kT) \\ -2.86\sin(2\pi 0.04kT) \\ 3.44\sin(2\pi 0.02kT) \end{bmatrix} \text{ deg/s}.$$

The sampling period, T, and the maximum rate gyros' noise, $\|n\|_\infty = \bar{n}$, are set to 0.1 s, and 0.115 deg/s, respectively. The initial

Table 1
RMS of the estimation errors of the SVO and of the MEKF (deg).

	x_{RMS}	y_{RMS}	z_{RMS}
SVO	0.0414	0.0318	0.0494
MEKF	0.0346	0.0350	0.0456

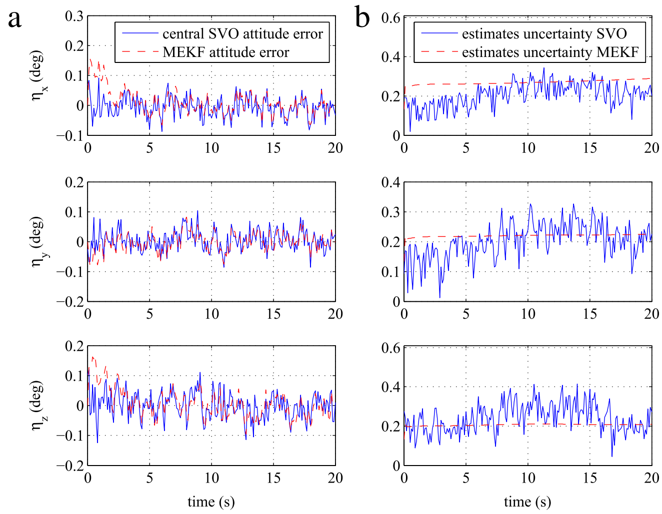

Fig. 1. Error of the central point of the rotation vector set and uncertainty limits along the main axis.

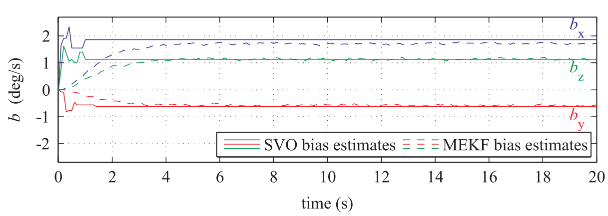

Fig. 2. Bias estimation errors.

uncertainty on the rate gyro bias is ±5.73 deg/s in each channel, while the true rate gyro bias is $b = [0.03\ -0.01\ 0.02]^T$ deg/s. The directions for the vector observations in the inertial frame are given by ${}^I v_1 = [1\ 4\ 0]^T$, ${}^I v_2 = [3\ 0\ 0]^T$, ${}^I v_3 = [0\ 0\ 6]^T$, and each channel of measurements is corrupted by noise bounded by ±0.01. The MEKF was initialized with the true initial attitude and the initial state covariance matrix was set to $P_0 = 6 \times 10^{-7} I_6$. The system covariance matrix was set to $Q = \frac{1}{4}\sigma_\omega^2 I_6$, and the measurements covariance matrix was set to $R = \sigma_v^2 I_9$, where σ_ω^2 and σ_v^2 are the variance of each axis of the angular velocity measurements and of the vector measurements, respectively.

The estimation errors of the central point of the obtained using the SVO and the MEKF are shown in Fig. 1(a). The estimates are expressed in exponential coordinates, computed by using (6). Both strategies have similar accuracies in terms of the root-mean-square (RMS) errors (Table 1).

Fig. 1(b) depicts the maximum uncertainty of the set-valued state estimates, as well as the 3σ bound of the MEKF state (based on the state covariance matrix). In this example, specially for the x and y axes, the bounds provided by the SVO are less conservative than the 3σ bounds of the MEKF. The estimates and the state covariance matrix of the MEKF are expressed in quaternions [8]. Thus, a transformation to exponential coordinates was necessary. The bias estimates provided by the SVO converged faster than those of the MEKF, as shown in Fig. 2.

When compared with the MEKF, the proposed technique presents some advantages: (i) it is global, (ii) it has faster convergence rates (at least in the simulations performed), (iii) it is robust against different sensor noise characteristics as no other information except worst-case bounds on sensor noise and external disturbances is required, and, finally, (iv) it provides a set-valued state estimate that is guaranteed to contain the true attitude. The computational load of the operations over the set-valued estimates can, however, hinder its implementation in real-time systems with low computational power. The MEKF is less computational expensive, although it may lead to degraded performance and, ultimately, to divergent solutions, due to the approximation of the dynamics of the system by a linear model.

6. Conclusions

This paper proposed a solution for the problem of attitude and rate gyro bias estimation, based on vector observations and angular velocity measurements, corrupted by bounded noise. We developed an SVO that considers uncertainties defined by polytopes and that, for any initial conditions, guarantees that the actual state of the system is inside the set-valued estimate. Conditions for boundedness of the estimated set were established. No linearization is required and the attitude of the rigid body is parameterized by a rotation matrix yielding estimates that are free of singularities. Several implementation details were discussed. Simulation results attested the applicability of the proposed technique.

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Appendix A. Proof of Lemma 1

Let $\omega(t) = \omega(t_0) + \omega_\Delta(t)$ and, as in [30], define $\varphi(t, t_0) = \mathrm{e}^{-t(\omega_0)^\times} \Phi(t, t_0)\, \mathrm{e}^{t_0(\omega_0)^\times}$, $\omega_0 = \omega(t_0)$, which fulfills $\dot{\varphi}(t, t_0) = \mathrm{e}^{-t(\omega_0)^\times} \omega_\Delta(t)\, \mathrm{e}^{t(\omega_0)^\times} \varphi(t, t_0)$. Then, a solution for $\varphi(t, t_0)$ can be obtained by resorting to the Peano–Baker series $\varphi(t, t_0) = \sum_{i=0}^{\infty} \varphi_i(t, t_0)$, $\varphi_0(t, t_0) = I_3$, $\varphi_i(t, t_0) = \int_{t_0}^{t} \mathrm{e}^{-\tau(\omega_0)^\times} \omega_\Delta(\tau) \mathrm{e}^{\tau(\omega_0)^\times} \varphi_{i-1}(\tau, t_0)\, \mathrm{d}\tau$. Define $\Phi_i(t, t_0) = \mathrm{e}^{t(\omega_0)^\times} \varphi_i(t, t_0)\, \mathrm{e}^{-t_0(\omega_0)^\times}$. Then, the state transition matrix satisfies $\Phi(t, t_0) = \Phi_0(t, t_0) + \Phi_\Delta(t, t_0)$, where

$$\Phi_0(t, t_0) = \mathrm{e}^{(t-t_0)(\omega_0)^\times}, \qquad \Phi_\Delta(t, t_0) = \sum_{i=1}^{\infty} \Phi_i(t, t_0),$$

$$\Phi_i(t, t_0) = \int_{t_0}^{t} \mathrm{e}^{(t-\tau)(\omega_0)^\times} \omega_\Delta(\tau) \Phi_{i-1}(\tau, t_0)\, \mathrm{d}\tau.$$

Algebraic manipulations allow us to obtain the relation $\|\Phi_i(t, t_0)\|_2 \le \int_{t_0}^{t} (t - t_0) \|\bar{\alpha}\|_2 \|\Phi_{i-1}(\tau, t_0)\|_2\, \mathrm{d}\tau \le \bar{\alpha}^i (t - t_0)^{2i} / (2^i i!)$. Thus, $\|\Phi_\Delta(t, t_0)\|_2 \le \mathrm{e}^{\bar{\alpha}(t-t_0)^2/2} - 1$. To conclude the proof, let $t_0 = kT$, $t = (k + 1)T$ and recall the relation between matrix norms, $\|A\|_{\max} \le \|A\|_2$.

Appendix B. Bound on the exponential map of the sum of two skew-symmetric matrices

Let $k_1 \in \mathbb{R}^3$ and $k_2 \in \mathbb{R}^3$ be two generic vectors and define the skew-symmetric matrices $K_1 = (k_1)^\times$, $K_2 = (k_2)^\times$. From the definition of matrix multiplication, we have that $[C]_{ij} = \sum_{k=1}^{p} [K_1]_{ik} [K_2]_{kj}$, where $C = K_1 K_2 \in \mathbb{R}^{3\times3}$, and $[X]_{ij}$ denotes the element of row i and column j of matrix $X \in \mathbb{R}^{m\times n}$. Using the fact that, for skew-symmetric matrices, at least one of the elements of each row and each column is zero, we obtain the following inequalities $\|K_1^k\|_{\max} \leq (2\bar{k}_1)^k/2$, $\|K_1^k K_2^l\|_{\max} \leq (2\bar{k}_1)^k(2\bar{k}_2)^l/2$, where $\bar{k}_1 = \|k_1\|_\infty$ and $\bar{k}_2 = \|k_2\|_\infty$. From these inequalities, we derive an upper bound for each element of the power of the sum of two matrices $\left[(K_1+K_2)^k\right]_{ij} \leq \left[K_1^k + \frac{1}{2}1_3((\bar{k}_1+\bar{k}_2)^k - \bar{k}_1^k)\right]_{ij}$, where 1_3 denotes a 3×3 matrix of ones. Consequently, the exponential map of the sum of matrices K_1 and K_2 satisfies $\left[e^{K_1+K_2}\right]_{ij} = \sum_{k=0}^{\infty} \left[(K_1+K_2)^k\right]_{ij}/k! \leq \left[e^{K_1}\right]_{ij} + \frac{1}{2}(e^{2\bar{k}_1+2\bar{k}_2} - e^{2\bar{k}_1})$.

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