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POSITIVE-DEFINITENESS, INTEGRAL EQUATIONS AND FOURIER TRANSFORMS

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ABSTRACT. We show that positive definite kernel functions $k(x, y)$, if continuous and integrable along the main diagonal, coincide with kernels of positive integral operators in $L^2(\mathbf{R})$. Such an operator is shown to be compact; under the further assumption $k(x, x) \to 0$ as $|x| \to \infty$ it is also trace class and the corresponding bilinear series converges absolutely and uniformly. If $k^{1/2}(x, x) \in L^1(\mathbf{R})$, all these results are carried through to a 'rotated' Fourier transform: $\hat{k}(\nu_1, -\nu_2)$ is the kernel of a compact positive operator and is represented by the absolutely and uniformly convergent series of Fourier transforms of eigenfunctions. The trace of the operator is an invariant under Fourier transforms.

1. Introduction. A number of recent applications renewed interest in the study of 'positive definite matrices' in the sense of Moore or, as we shall call them below, a positive definite kernel functions. In signal processing many physical phenomena are modeled by random processes; for second order processes, reconstruction of the signal by sampling requires consideration of the autocorrelation function both in the time and frequency domains. This function is by construction a positive definite kernel function [**3**]. In a similar vein, the theory of machine learning leads to similar questions [**4**]. It thus becomes a problem of interest for applications to study this class of functions and their Fourier transforms.

The aim of this paper is to carry out this study. We show in Section 3 that, under the assumptions of continuity and summability along the diagonal, a positive definite kernel function $k(x, y)$ is the kernel of a positive integral operator in $L^2(\mathbf{R})$. We show that positivity implies that this operator is Hilbert-Schmidt and thus necessarily compact. It then follows from standard spectral theory that k is expressed by an L^2

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convergent bilinear eigenfunction series. Under the further assumption that $k(x,x) \to 0$ as $|x| \to 0$, we show that this series is absolutely and uniformly convergent in $\mathbf{R}$ and the associated operator is trace-class with $\operatorname{tr} \mathcal{K} = \int_{-\infty}^{+\infty} k(x,x)\, dx = \sum_{i \geq 0} \mu_i$. This differs from the classical Mercer's theorem in that the domain of the operator is not compact.

In Section 4 we show that many of these properties carry through Fourier transformation. Defining $\tilde{k}(\nu_1, \nu_2) = \hat{k}(\nu_1, -\nu_2)$, we show that k is the kernel of a positive operator if and only if $\tilde{k}$ is the kernel of a positive operator. Moreover, these operators have the same eigenvalues, and the corresponding orthonormal eigenfunctions are the Fourier transforms of each other and $\tilde{k}$ has an L^2 convergent bilinear eigenfunction expansion. Under the further assumptions that $k(x,x) \to 0$ as $|x| \to 0$ and $k^{1/2}(x,x) \in L^1(\mathbf{R})$, this expansion converges absolutely and uniformly and the Fourier transformed operator $\tilde{\mathcal{K}}$ is a trace class with

$$\operatorname{tr} \tilde{\mathcal{K}} = \operatorname{tr} \mathcal{K} = \int_{-\infty}^{+\infty} k(x,x)\, dx = \int_{-\infty}^{+\infty} \tilde{k}(\nu,\nu)\, d\nu = \sum_{i \geq 0} \mu_i,$$

so that the trace may be thought of as a Fourier invariant for this class of operators.

2. Positive definite kernel functions and kernels. We now introduce the classes of functions with which we will work throughout this paper, namely positive definite kernel functions and L^2-positive definite kernels.

Definition 2.1. Let $k : \mathbf{R}^2 \to \mathbf{C}$. We say that k is a *positive definite kernel function* (PDKF) if

$$\sum_{i,j=1}^{n} k(x_i, x_j)\, \overline{z_i}\, z_j \geq 0 \tag{1}$$

for all $n \in \mathbf{N}$, $(x_1, \cdots, x_n) \in \mathbf{R}^n$ and $(z_1, \cdots, z_n) \in \mathbf{C}^n$.

Remark 2.2. Positive definite kernel functions were first introduced by Moore, see e.g., Aronszajn [**1**] under the name 'positive definite matrices'. The term *positive definite* stems naturally from the analogy with the finite (matrix) case. To avoid confusion we avoid this terminology. We use the term *kernel* in view of the link between positive definite kernel functions and the L^2-positive definite kernels (L^2-PDK) to be introduced below.

On the other hand, positive definite kernel functions are related to the widely studied class of *positive definite functions* introduced independently by Mathias, Caratheodory and Bochner. These are functions $f : \mathbf{R} \to \mathbf{C}$ which satisfy (1) with $k(x_i, x_j)$ replaced with $f(x_i - x_j)$. The most important result regarding positive definite functions is Bochner's theorem, which characterizes positive definite functions as the Fourier-Stieltjes transform of positive measures; see e.g. Stewart [**10**] and Rudin [**8**]. It is also to avoid confusion with these that we choose the term PDKF.

Example 2.3. A first example of a positive definite kernel function is $k(x, y) = \phi(x)\overline{\phi(y)}$, where $\phi : \mathbf{R} \to \mathbf{C}$. A less trivial example is the following: given a set $\{\phi_n(x)\}_{n\geq 0}$ and a convergent series $\sum_{n=0}^{\infty} \mu_n$ of nonnegative terms, i.e. $\mu_n \geq 0$, suppose that the series $\sum_{n\geq 0} \mu_n\phi_n(x)\overline{\phi_n(y)}$ is convergent for every $(x, y) \in \mathbf{R}^2$. Then, from Remark 2.4 below we conclude that

$$k(x, y) = \sum_{n\geq 0} \mu_n\phi_n(x)\overline{\phi_n(y)} \tag{2}$$

is a PDKF. In fact, we will show in Section 3 that any continuous PDKF k with $k(x, x) \in L^1(\mathbf{R})$ is of the form (2), where $\{\phi_n\}_{n\geq 0}$ is an L^2-orthonormal set of continuous functions and the series is L^2 convergent.

Remark 2.4. The following properties of PDKFs are easy to verify. First of all, if k is a PDKF, then so is $\bar{k}$. Secondly, if $k_1, k_2, \ldots, k_n$ are PDKFs and $c_i \geq 0$ for $1 \leq i \leq n$, then $\sum_{i=1}^{n} c_i k(x, y)$ is a PDKF. This implies that the set of PDKFs forms a cone in the space of all functions from $\mathbf{R}^2$ to $\mathbf{C}$; it is easy to see that the relation $\leq$ induces a partial ordering in this set. Finally, if $\{k_n\}_{n\geq 0}$ is a sequence of PDKFs converging pointwise to k, then k is a PDKF.

In the following proposition we state the most important properties of PDKFs for our purposes. Properties (a), (b) and (c) are well-known from the literature, following easily from well-known facts about 'positive definite matrices' in the sense of Moore. Their proof is omitted.

Proposition 2.5. *Suppose* $k : \mathbf{R}^2 \to C$ *is a* PDKF. *Then*:

(a) *k is positive on the diagonal $x_1 = x_2$, that is, $k(x,x)$ is real and greater than or equal to* 0 *for all $x \in \mathbf{R}$.*

(b) *for all $x_1, x_2 \in \mathbf{R}$ $k(x_1,x_2) = \overline{k(x_2,x_1)}$.*

(c) *for all $x_1, x_2 \in \mathbf{R}$, $|k(x_1,x_2)|^2 \le k(x_1,x_1)\, k(x_2,x_2)$.*

(d) *If $k(x,x) \in L^1(\mathbf{R})$, then $k(x_1,x_2) \in L^2(\mathbf{R}^2)$ and $\|k\|_{L^2} \le \left(\int_{-\infty}^{+\infty} k(x,x)\, dx\right)^2$.*

Proof. As stated, (a), (b) and (c) are well-known from the literature; see, e.g., Aronszajn [**1**] or Cucker and Smale [**4**].

To prove (d), note that by (c) above we have that for all $x_1, x_2 \in \mathbf{R}$, $|k(x_1,x_2)|^2 \le k(x_1,x_1)\, k(x_2,x_2)$. Thus

$$\begin{aligned}\iint_{\mathbf{R}^2} |k(x_1,x_2)|^2\, dx_1\, dx_2 &\le \iint_{\mathbf{R}^2} k(x_1,x_1) k(x_2,x_2)\, dx_1\, dx_2 \\ &= \left(\int_{-\infty}^{+\infty} k(x,x)\, dx\right)^2\end{aligned}$$

showing that since k is positive on the diagonal $x_1 = x_2$ by (a), then $k \in L^2(\mathbf{R}^2)$ if $k(x,x) \in L^1(\mathbf{R})$. □

Remark 2.6. It is straightforward to note that, replacing $k(x_i, x_j)$ with $f(x_i - x_j)$, the most important properties of positive definite functions can be recovered from Proposition 2.5. In fact, if $f : \mathbf{R} \to C$ we conclude immediately that

a) $f(0)$ is real and ≥ 0;

b) $f(-x) = \overline{f(x)}$ for all $x \in \mathbf{R}$;

c) $|f(x)| \le f(0)$ for all $x \in \mathbf{R}$, implying in particular that f is bounded.

We now introduce the integral analog of Definition 2.1.

Definition 2.7. Let $k \in L^2(\mathbf{R}^2)$. We say that k is an L^2-*positive definite kernel* (L^2-PDK) if

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} k(x,y)\, \phi(y)\overline{\phi(x)}\, dx\, dy \geq 0 \tag{3}$$

for all $\phi \in L^2(\mathbf{R})$.

Remark 2.8. A close link exists between conditions (1) and (3). With particular relevance for our purposes is the fact that, if k is a continuous $L^2(\mathbf{R}^2)$ function, then k is a PDKF if and only if it is an L^2-PDK, and thus for continuous L^2 kernels conditions (1) and (3) are equivalent. This was first shown in the compact case by Mercer, see Stewart [**10**]; Rudin's proof [**8**] of this fact for positive definite functions translates immediately to our setting and we omit it.

We should note that dropping the L^2 requirement changes the picture drastically. For continuous functions (3) is much stronger than (1): as our first example in 2.3 shows, a PDKF need not be bounded or even measurable.

Remark 2.9. An immediate consequence of Remark 2.8 is that, if k is a continuous L^2-PDK, then k has all the properties described in Proposition 2.5.

Remark 2.10. It is useful to look at Definition 2.7 from the point of view of operator theory. Given $k \in L^2(\mathbf{R}^2)$ we define a linear operator $\mathcal{K} : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ by setting

$$\phi \longmapsto \mathcal{K}(\phi) = \int_{-\infty}^{+\infty} k(x,y)\, \phi(y)\, dy,$$

that is, as an integral operator in $\mathbf{R}$ with kernel k.

The following facts are a consequence of standard operator theory, see e.g. Reed and Simon [**6**]. First of all, the fact that $k(x_1,x_2) = \overline{k(x_2,x_1)}$ from Proposition 2.5 and Remark 2.8 implies that $\mathcal{K}$ is self-adjoint. Secondly, Proposition 2.5 implies that $\|\mathcal{K}\| = \|k\|_{L^2} \leq \left(\int_{-\infty}^{+\infty} k(x,x)\, dx\right)^2$; therefore $\mathcal{K}$ is Hilbert-Schmidt, and thus an automatically compact, operator. Finally, equation (3) states that $\int_{-\infty}^{+\infty} \mathcal{K}(\phi)(x)\,\overline{\phi(x)}\,dx \geq 0$ or, in terms of the standard inner product in $L^2(\mathbf{R})$, $\langle \mathcal{K}(\phi), \phi \rangle \geq 0$. Thus $\mathcal{K}$ is a positive operator.

These properties imply the following spectral properties of $\mathcal{K}$: the spectrum is formed by a finite or countable sequence of real eigenvalues $\{\mu_n\}_{n\geq 0}$ with $\mu_n \geq 0$ whose only possible limit point is 0; the multiplicity of every nonzero eigenvalue is finite; and $L^2(\mathbf{R})$ has a complete orthonormal basis $\{\phi_n\}_{n\geq 0}$, where the $\{\phi_n\}$ are eigenfunctions of $\mathcal{K}$ (Hilbert-Schmidt theorem).

In the rest of the paper we suppose for convenience that the eigenvalues have been ordered so that $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \ldots \geq 0$.

3. Characterization of L^2-PDKs. In this section we show that continuous PDKFs such that $k(x,x) \in L^1(\mathbf{R})$ are necessarily L^2-positive definite kernels, with all the resulting consequences. In the rest of the paper we thus restrict to continuous L^2-PDKs. We provide a spectral characterization of the class of continuous L^2-PDKs such that $k(x,x) \in L^1(\mathbf{R})$ and $k(x,x) \to 0$ as $|x| \to \infty$, showing in particular that the bilinear series expansion is uniformly and absolutely convergent and that the corresponding integral operator is trace class. We also show that the stronger conditions $k^{1/2}(x,x) \in L^1(\mathbf{R})$ and $k(x,x) \to 0$ as $|x| \to \infty$ imply sharp L^1 norm estimates for k and for the eigenfunctions associated with nonzero eigenvalues.

Proposition 3.1. *Let $k : \mathbf{R}^2 \to \mathbf{C}$ be a continuous* PDKF *such that $k(x,x) \in L^1(\mathbf{R})$. Then k is a continuous L^2-positive definite kernel and therefore the kernel of an L^2 positive integral operator $\mathcal{K}$.*

Proof. From Proposition 2.5 the condition $k(x,x) \in L^1(\mathbf{R})$ implies that $k(x,y) \in L^2(\mathbf{R}^2)$. From Remark 2.10 it follows that $\mathcal{K}$ is Hilbert-Schmidt and thus compact. Since k is continuous, from Remark 2.8 we conclude that k is an L^2-positive definite kernel, that is, the associated L^2 integral operator

$$\mathcal{K}(\phi) = \int_{-\infty}^{+\infty} k(x,y)\,\phi(y)\,dy$$

is positive. □

Corollary 3.2. *Let $k : \mathbf{R}^2 \to \mathbf{C}$ be a continuous* PDKF *such that $k(x,x) \in L^1(\mathbf{R})$. Then k admits the bilinear series expansion*

$$k(x,y) = \sum_{i \geq 0} \mu_i \, \phi_i(x) \, \overline{\phi_i(y)}, \tag{4}$$

where the $\{\phi_i\}_{i \geq 0}$ are the L^2-orthonormal eigenfunctions of the associated positive integral operator $\mathcal{K}$, μ_i are the eigenvalues of $\mathcal{K}$ and the bilinear series (4) *converges in L^2.*

Proof. The existence of the bilinear expansion with these properties is a consequence of Proposition 3.1 and standard facts from Hilbert space operator theory, see e.g., [**6, 7**]. ◻

Remark 3.3. We may interpret these results as stating that the extra condition of integrability along the diagonal forces the class of continuous PDKFs to coincide with the class of continuous L^2-PDKs. We shall henceforth, without loss of generality, formulate all our results in terms of the latter.

Theorem 3.4. *Let $k : \mathbf{R}^2 \to \mathbf{C}$ be a continuous L^2-*PDK *such that $k(x,x) \in L^1(\mathbf{R})$ and $k(x,x) \to 0$ as $|x| \to \infty$. Then*

(i) *Eigenfunctions ϕ_i of the associated operator $\mathcal{K}$ associated with nonzero eigenvalues are uniformly continuous vanishing at infinity.*

(ii) *The bilinear series* (4) *converges absolutely and uniformly to k.*

(iii) *The operator $\mathcal{K}$ is trace class with*

$$\operatorname{tr} \mathcal{K} = \int_{-\infty}^{+\infty} k(x,x) \, dx = \sum_{i \geq 0} \mu_i. \tag{5}$$

Proof. We first prove (i). Since $k(x,x) \in L^1(\mathbf{R})$ and is continuous, the assumption $k(x,x) \to 0$ as $|x| \to \infty$ implies that $k(x,x)$ is uniformly continuous in $\mathbf{R}$.

On the other hand, we note that $\lim_{x \to x_0} \int_{-\infty}^{+\infty} |k(x_0,y) - k(x,y)|^2 \, dy = 0$ for all $x_0 \in \mathbf{R}$ implies that, for all $\phi \in L^2(\mathbf{R})$, $\mathcal{K}(\phi)(x)$ is a continuous function. In fact,

(6)

$$
\begin{aligned}
|\mathcal{K}(\phi)(x_0) - \mathcal{K}(\phi)(x)| &= \left| \int_{-\infty}^{+\infty} k(x_0, y)\, \phi(y)\, dy - \int_{-\infty}^{+\infty} k(x, y)\, \phi(y)\, dy \right| \\
&\le \int_{-\infty}^{+\infty} |(k(x_0, y) - k(x, y))|\, |\phi(y)|\, dy \\
&\le \left[\int_{-\infty}^{+\infty} |k(x_0, y) - k(x, y)|^2\, dy \right]^{1/2} \|\phi\|_{L^2},
\end{aligned}
$$

from which the statement follows by taking limits. In particular eigenfunctions of $\mathcal{K}$ are continuous.

It is shown elsewhere [**2**] that for continuous $L^2(\mathbf{R}^2)$-PDKs the condition

$$
\lim_{x \to x_0} \int_{-\infty}^{+\infty} |k(x, y) - k(x_0, y)|^2\, dy = 0
$$

is automatically satisfied. This property together with summability and uniform continuity of $k(x, x)$ allow us to apply Novitskii's generalization of Mercer's theorem to unbounded domains [**5**]. We thus conclude that

$$
k(x, y) = \sum_{i \ge 0} \mu_i\, \phi_i(x)\, \overline{\phi_i(y)},
$$

where the $\phi_i(x)$ are the eigenfunctions of the associated integral operator $\mathcal{K}$, which are continuous and L^2-orthonormal; the μ_i are the corresponding eigenvalues which by compactness and positivity of $\mathcal{K}$ are real, positive and have 0 as its only possible limit point; and the bilinear series above converges absolutely and uniformly in $\mathbf{R}^2$. Uniform continuity of the eigenfunctions associated with nonzero eigenvalues follows from the fact that $k(x, x) \ge \sum_{n=1}^{N} \mu_i\, |\phi_i(x)|^2$. Since $k(x, x) \to 0$ as $|x| \to \infty$ it follows that, for every i, $|\phi_i(x)| \to 0$ as $|x| \to \infty$. Since ϕ_i is continuous, ϕ_i is uniformly continuous, proving (ii).

To prove (iii), we note that since the convergence of the series (4) is uniform, term by term integration is permissible, yielding $\int_{-\infty}^{+\infty} k(x, x) \times dx = \sum_{i \ge 0} \mu_i$. The set $\{\phi_i\}_{i \ge 0}$ of eigenfunctions associated with nonzero eigenvalues forms a complete orthonormal basis for the range of $\mathcal{K}$. To these we adjoin a complete orthonormal basis for the null space of $\mathcal{K}$; by the Hilbert-Schmidt theorem, we obtain a complete orthonormal basis for $L^2(\mathbf{R})$. Computing the trace of $\mathcal{K}$ in this basis yields

$$\operatorname{tr}\mathcal{K} = \sum_{i\geq 0} \langle \mathcal{K}\phi_i, \phi_i\rangle = \sum_{i\geq 0} \mu_i \langle \phi_i, \phi_i\rangle = \sum_{i\geq 0} \mu_i,$$

since eigenfunctions associated with 0 do not contribute to the sum. This completes the proof. □

Remark 3.5. The hypothesis $k(x,x) \to 0$ as $|x| \to \infty$, which implies uniform continuity of $k(x,x)$, is essential in Theorem 3.4, as the following counterexample shows. Choose $\mu_n = 1/n^2$. It is not hard to construct a family of continuous functions $\{\phi_n(x)\}$ such that $\|\phi_n\|_{L^2} = 1$ for all $n \in \mathbf{N}$, the support of ϕ_n is contained in $I_n = [n, n+1]$, $\max_{I_n} \phi_n > n^2$ and $\sum_{n\geq 0} \mu_n |\phi_n(x)|^2 \in L^1(\mathbf{R})$. Then the associated operator satisfies all other conditions in Theorem 3.4 but the series (4) does not converge uniformly since each finite sum is bounded but the infinite sum is not.

With slightly stronger hypotheses we can derive a version of Theorem 3.4 which shows how it is related to the rate of decay of the kernel along the diagonal.

Corollary 3.6. *Let $\mathcal{K} : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ be a positive integral operator with continuous kernel $k(x,y)$ satisfying $k(x,x) = O(1/x^{1+\varepsilon})$ for some $\varepsilon > 0$. Then all the statements of Theorem* 3.4 *hold.*

Proof. $k(x,x) = O(1/x^{1+\varepsilon})$ implies the hypotheses of Theorem 3.4. □

In the following lemma we use the positive function $k^{1/2}(x,x)$, which is well-defined and continuous since $k(x,x) \geq 0$ and is continuous for all $x \in \mathbf{R}$.

Lemma 3.7. *Let $\mathcal{K} : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ be a positive integral operator with continuous kernel $k(x,y)$ satisfying $k^{1/2}(x,x) \in L^1(\mathbf{R})$ and $k(x,x) \to 0$ as $|x| \to \infty$. Then, in addition to the statements of*

Theorem 3.4:

(i) *For all* $\phi \in L^2(\mathbf{R})$, $\mathcal{K}\phi(x) \in L^1(\mathbf{R})$ *and*

$$\|\mathcal{K}\phi\|_{L^1} \leq \int_{-\infty}^{+\infty} k^{1/2}(x,x)\,dx \left[\int_{-\infty}^{+\infty} k(x,x)\,dx\right]^{1/2} \|\phi\|_{L^2}.$$

(ii) *Eigenfunctions* ϕ_i *associated to nonzero eigenvalues* μ_i *of* $\mathcal{K}$ *are in* $L^1(\mathbf{R})$ *and*

$$\|\phi_i\|_{L^1} \leq \frac{1}{\mu_i} \int_{-\infty}^{+\infty} k^{1/2}(x,x)\,dx \left[\int_{-\infty}^{+\infty} k(x,x)\,dx\right]^{1/2} \|\phi_i\|_{L^2}.$$

(iii) $k(x,y) \in L^1(\mathbf{R}^2)$ *with* L^1 *norm bounded by*

$$\|k\|_{L^1(\mathbf{R}^2)} \leq \left[\int_{-\infty}^{+\infty} k^{1/2}(x,x)\,dx\right]^2,$$

and the bilinear series (4) *converges to* k *in the* L^1 *norm.*

Proof. The hypotheses on k imply those of Theorem 3.4: $k^{1/2}(x,x) \in L^1(\mathbf{R})$ and $k(x,x) \to 0$ as $|x| \to \infty$ imply $k(x,x) \in L^1(\mathbf{R})$. Thus all the corresponding results hold.

Given $\phi \in L^2(\mathbf{R})$ we estimate the L^1 norm of $\mathcal{K}\phi$ as follows:

$$\begin{aligned}
\|\mathcal{K}\phi\|_{L^1} &= \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} k(x,y)\phi(y)\,dy \right| dx \\
&\leq \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |k(x,y)|\,|\phi(y)|\,dy\right] dx \\
&\leq \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |k(x,y)|^2\,dy\right]^{1/2} \left[\int_{-\infty}^{+\infty} |\phi(y)|^2\,dy\right]^{1/2} dx \\
&\leq \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} k(x,x)k(y,y)\,dy\right]^{1/2} \left[\int_{-\infty}^{+\infty} |\phi(y)|^2\,dy\right]^{1/2} dx \\
&= \int_{-\infty}^{+\infty} k^{1/2}(x,x)\,dx \left[\int_{-\infty}^{+\infty} k(y,y)\,dy\right]^{1/2} \|\phi\|_{L^2},
\end{aligned}$$

proving statement (i).

If ϕ_i is an eigenfunction of $\mathcal{K}$ associated with a nonzero eigenvalue μ_i, that is, $\mathcal{K}(\phi_i) = \mu_i \phi_i$, by the previous paragraph $\phi_i \in L^1(\mathbf{R})$ with L^1 norm satisfying

$$\|\phi_i\|_{L^1} \leq \frac{1}{\mu_i} \int_{-\infty}^{+\infty} k^{1/2}(x,x)\, dx \left[\int_{-\infty}^{+\infty} k(y,y)\, dy \right]^{1/2} \|\phi_i\|_{L^2}, \tag{7}$$

proving (ii). To prove (iii), we calculate the $L^1(\mathbf{R}^2)$ norm of $k(x,y)$:

$$\begin{aligned}
\|k\|_{L^1(\mathbf{R}^2)} &= \iint_{\mathbf{R}^2} |k(x,y)|\, dx\, dy \\
&\leq \iint_{\mathbf{R}^2} k^{1/2}(x,x) k^{1/2}(y,y)\, dx\, dy \\
&= \int_{-\infty}^{+\infty} k^{1/2}(x,x)\, dx \int_{-\infty}^{+\infty} k^{1/2}(y,y)\, dy \\
&= \left[\int_{-\infty}^{+\infty} k^{1/2}(x,x)\, dx \right]^2 < +\infty.
\end{aligned}$$

Thus $k \in L^1(\mathbf{R}^2)$. Since $\left[\sum_{n\geq 0} \mu_n |\phi_n(x)|^2\right]^{1/2}$ is monotonely convergent to $k^{1/2}(x,x) \in L^1(\mathbf{R})$, it follows from Lebesgue's theorem that

$$\int_{-\infty}^{+\infty} \left[\sum_{n\geq 0} \mu_n |\phi_n(x)|^2 \right]^{1/2} dx = \int_{-\infty}^{+\infty} k^{1/2}(x,x)\, dx,$$

thus proving that the bilinear series converges in the L^1 norm. By Theorem 3.4 its limit is of course $k(x,y)$ for all $x, y \in \mathbf{R}$, providing the bound $\|k\|_{L^1} \leq \left[\int_{-\infty}^{+\infty} k^{1/2}(x,x)\, dx\right]^2$. □

As with Theorem 3.4, a slightly weaker version of this result shows how it relates to the rate of decay of the kernel along the diagonal.

Corollary 3.8. *Let $\mathcal{K} : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ be a positive integral operator with continuous kernel $k(x,y)$ satisfying $k(x,x) = O(1/x^{2+\epsilon})$ for some $\varepsilon > 0$. Then all the statements of Theorem* 3.4 *and Lemma* 3.7 *hold.*

Proof. $k(x,x) = O(1/x^{2+\varepsilon})$ implies trivially $k(x,x) \to 0$ as $|x| \to \infty$ and $k(x,x) \in L^1(\mathbf{R})$ while $k^{1/2}(x,x) = O(1/x^{1+\varepsilon/2})$ implies $k^{1/2}(x,x) \in L^1(\mathbf{R})$. □

Remark 3.9. Lemma 3.7 is sharp, as the following counterexample shows. Let $\phi(x) = \sin x/x$ and consider the positive operator with kernel $k(x,y) = \phi(x)\phi(y)$. This operator is of finite rank; indeed it has a single simple nonzero eigenvalue π, with associated normalized eigenfunction $1/\sqrt{\pi}\phi(x)$. Obviously $k(x,x) = \sin^2 x/x^2$ is in $L^1(\mathbf{R})$ and $k(x,x) \to 0$ as $|x| \to \infty$. On the other hand, $k^{1/2}(x,x) \in L^p(\mathbf{R})$ for every $p > 1$ but $k^{1/2}(x,x) \notin L^1(\mathbf{R})$. In this case all the statements of Lemma 3.7 are false: $\mathcal{K}\phi(x) = \pi\phi(x) \notin L^1(\mathbf{R})$, so that neither the image of L^2 is contained in L^1 nor the eigenfunctions associated with nonzero eigenvalues are in L^1. Furthermore, $k(x,y)$ itself is not in $L^1(\mathbf{R}^2)$, and therefore the bilinear series (which in this case trivially has a single term) is not L^1-convergent.

4. Fourier transforms of L^2-PDKs. In this section we show that an L^2 kernel is positive definite if and only if its 'rotated' Fourier transform $\hat{k}(\nu_1, -\nu_2)$ is an L^2-positive definite kernel with the same eigenvalues and whose eigenfunctions are the Fourier transforms of the original eigenfunctions. We provide sharp sufficient conditions along the diagonal of an L^2-PDK to ensure that the properties of uniform convergence and being trace class carry through the Fourier transformation. In particular the trace of the associated integral operator is, in this sense, invariant under Fourier transforms.

We consider below Fourier transforms of positive definite kernels. We will use throughout the L^2 version of the Fourier transform, briefly recalling some facts to be used; a standard reference is, e.g., Stein and Weiss [**9**].

For $f \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ one may define the Fourier transform $\hat{f}$ of f by

$$\hat{f}(\boldsymbol{\nu}) = \int_{\mathbf{R}^n} f(\mathbf{t}) e^{-2\pi i \boldsymbol{\nu}.\mathbf{x}}\, d\mathbf{x}, \tag{8}$$

where $\boldsymbol{\nu}.\mathbf{x}$ denotes the usual inner product in $\mathbf{R}^n$. This may be extended by density to $L^2(\mathbf{R}^n)$, yielding a linear map $\mathcal{F} : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ which is an L^2 isometry and a Hilbert space isomorphism. The inverse Fourier transform is likewise defined in $L^2(\mathbf{R}^n)$; if $f \in L^2(\mathbf{R}^n)$ and $\hat{f} \in L^1(\mathbf{R}^n)$ it is given by the inversion formula

$$f(\mathbf{x}) = \int_{\mathbf{R}^n} \hat{f}(\boldsymbol{\nu}) e^{2\pi i \boldsymbol{\nu}.\mathbf{x}}\, d\boldsymbol{\nu} \quad \text{a.e.,} \tag{9}$$

with equality holding everywhere if f is continuous. Finally, for all $f, g \in L^2(\mathbf{R})$ the following Parseval identity holds:

$$\langle f, g\rangle = \int_{\mathbf{R}^n} f(\mathbf{x})\, \overline{g(\mathbf{x})}\, d\mathbf{x} = \int_{\mathbf{R}^n} \hat{f}(\boldsymbol{\nu})\, \overline{\hat{g}(\boldsymbol{\nu})}\, d\boldsymbol{\nu}. \tag{10}$$

In this paper we shall be concerned mainly with the two-dimensional case, with occasional use of the case $n = 1$.

We now consider the implications of positive definiteness of $k \in L^2(\mathbf{R}^2)$ on its Fourier transform. That some such relationship should exist is suggested by the Bochner theorem for positive definite functions. In the $L^2(\mathbf{R}^2)$ setting we prove that positive definiteness is a condition that carries through Fourier transforms in a symmetric way: namely, a function is an L^2-positive definite kernel *if and only if* its double Fourier transform is itself (in an appropriate sense) an L^2-positive definite kernel with the same spectral properties. This is the content of the next results.

Proposition 4.1. *Let $k \in L^2(\mathbf{R}^2)$ and $\hat{k}$ be its Fourier transform. Then $k(x, y)$ is a positive definite kernel if and only if $\hat{k}(\nu_1, -\nu_2)$ is a positive definite kernel.*

Proof. For each $\phi \in L^2(\mathbf{R})$ define the functional $M_\phi : L^2(\mathbf{R}^2) \to \mathbf{C}$ by

$$M_\phi(k) = \iint_{\mathbf{R}^2} k(x, y)\, \phi(y)\overline{\phi(x)}\, dx\, dy. \tag{11}$$

We begin by showing that, for every $\phi \in L^2(\mathbf{R})$, $M_\phi(k) = M_{\hat{\phi}}(\tilde{k})$, where $\hat{\phi}$ is the Fourier transform of ϕ and $\tilde{k}(\nu_1, \nu_2) = \hat{k}(\nu_1, -\nu_2)$.

Suppose $k \in L^1(\mathbf{R}^2) \cap L^2(\mathbf{R}^2)$ and $\phi \in L^2(\mathbf{R})$. Using Fubini's theorem we may write $M_\phi(k) = \int_{-\infty}^{+\infty} \overline{\phi(x)} \left[\int_{-\infty}^{+\infty} k(x,y)\,\phi(y)\,dy\right] dx$. Noting that $k(x,y) \in L^2(\mathbf{R})$ as a function of y for almost every value of x, writing $\hat{k}(x,\nu_2)$ for the Fourier transform of k in the second variable and observing that $\hat{\overline{\phi}}(\nu) = \overline{\hat{\phi}}(-\nu)$, we get from Parseval's identity

$$M_\phi(k) = \int_{\mathbf{R}} \overline{\phi}(x) \left[\int_{\mathbf{R}} \hat{k}(x,\nu_2)\hat{\phi}(-\nu_2)\,d\nu_2\right] dx.$$

Since $k \in L^1(\mathbf{R}^2) \cap L^2(\mathbf{R}^2)$, we also have $k(x,y) \in L^1(\mathbf{R})$ as a function of y for almost every value of x. Hence we may write

$$M_\phi(k) = \int_{\mathbf{R}} \overline{\phi}(x) \int_{\mathbf{R}} \left[\int_{\mathbf{R}} k(x,y)e^{-2\pi i\nu_2 y}\,dy\right]\hat{\phi}(-\nu_2)\,d\nu_2\,dx;$$

using Fubini's theorem again we have

$$M_\phi(k) = \int_{\mathbf{R}} \hat{\phi}(-\nu_2) \int_{\mathbf{R}} e^{-2\pi i\nu_2 y} \int_{\mathbf{R}} k(x,y)\overline{\phi(x)}\,dx\,dy\,d\nu_2.$$

From Parseval's identity and performing a partial Fourier transform on the variable x we get, in the same way as we did for y,

$$\begin{aligned} M_\phi(k) &= \int_{\mathbf{R}} \hat{\phi}(-\nu_2) \int_{\mathbf{R}} e^{-2\pi i\nu_2 y} \int_{\mathbf{R}} \hat{k}(\nu_1,y)\overline{\hat{\phi}}(\nu_1)\,d\nu_1\,dy\,d\nu_2 \\ &= \int_{\mathbf{R}} \hat{\phi}(-\nu_2) \int_{\mathbf{R}} e^{-2\pi i\nu_2 y} \int_{\mathbf{R}} \left[\int_{\mathbf{R}} k(x,y)e^{-2\pi i\nu_1 x}\,dx\right] \\ &\qquad\qquad \times \overline{\hat{\phi}}(\nu_1)\,d\nu_1\,dy\,d\nu_2. \end{aligned}$$

By Fubini's theorem and the use of the definition of Fourier transform in $\mathbf{R}^2$, we finally obtain

$$\begin{aligned} M_\phi(k) &= \int_{\mathbf{R}}\int_{\mathbf{R}} \overline{\hat{\phi}}(\nu_1)\hat{\phi}(-\nu_2)\left[\int_{\mathbf{R}}\int_{\mathbf{R}} k(x,y)e^{-2\pi i\boldsymbol{\nu}.\mathbf{x}}\,dx\,dy\right] d\nu_1\,d\nu_2 \\ &= \iint_{\mathbf{R}^2} \hat{k}(\nu_1,\nu_2)\,\hat{\phi}(-\nu_2)\overline{\hat{\phi}}(\nu_1)\,d\nu_1\,d\nu_2 \\ &= \iint_{\mathbf{R}^2} \hat{k}(\nu_1,-\nu_2)\,\hat{\phi}(\nu_2)\overline{\hat{\phi}}(\nu_1)\,d\nu_1\,d\nu_2 \\ &= M_{\hat{\phi}}(\hat{k}(\nu_1,-\nu_2)) \\ &= M_{\hat{\phi}}(\tilde{k}). \end{aligned}$$

Now, let $k \in L^2(\mathbf{R}^2)$ and $k_n \in L^1(\mathbf{R}^2) \cap L^2(\mathbf{R}^2)$ be a sequence converging to k in the L^2 norm. Then $\tilde{k_n} \to \tilde{k}$ and, since M_ϕ is L^2-continuous, we get $M_\phi(k) = M_{\hat{\phi}}(\tilde{k})$.

To finish the proof we finally observe that, from (11), k is an L^2-PDK if and only if $M_\phi(k) \geq 0$ for every $\phi \in L^2(\mathbf{R})$ and that $\tilde{k}(\nu_1, \nu_2) = \hat{k}(\nu_1, -\nu_2)$ is an L^2-PDK if and only if $M_{\hat{\phi}}(\tilde{k}) \geq 0$ for every $\hat{\phi} \in L^2(\mathbf{R}^2)$. Because the Fourier transform is an L^2 isomorphism, $\hat{y}$ ranges over $L^2(\mathbf{R})$ when y ranges over $L^2(\mathbf{R})$, and the equality $M_\phi(k) = M_{\hat{\phi}}(\tilde{k})$ establishes the result. □

For ease of reference we use from now on the notation $\tilde{k}(\nu_1, \nu_2) = \hat{k}(\nu_1, -\nu_2)$ and denote by $\tilde{\mathcal{K}}$ the positive $L^2(\mathbf{R}^2)$ operator with kernel $\tilde{k}(\nu_1, \nu_2)$.

Proposition 4.2. *Let $k \in L^2(\mathbf{R}^2)$ be an L^2-*PDK *and $\tilde{k}(\nu_1, \nu_2) = \hat{k}(\nu_1, -\nu_2)$. Then*:

(i) *The operators $\mathcal{K}$ and $\tilde{\mathcal{K}}$ have the same spectra, that is, the same eigenvalues with the same multiplicities.*

(ii) *ϕ_i is an eigenfunction of $\mathcal{K}$ associated with the eigenvalue μ_i if and only if $\hat{\phi}_i$ is an eigenfunction of $\tilde{\mathcal{K}}$ associated with the eigenvalue μ_i.*

(iii) *k is given by an L^2 bilinear expansion* (4) *if and only if*

$$\tilde{k}(\nu_1, \nu_2) = \sum_{i \geq 0} \mu_i \, \hat{\phi}_i(\nu_1) \, \overline{\hat{\phi}_i(\nu_2)},$$

where the $\hat{\phi}_i$ are the Fourier transforms of the ϕ_i and the convergence properties are the same as in (4).

Proof. $\phi_i \in L^2(\mathbf{R})$ is an eigenfunction of $\mathcal{K}$ associated with the eigenvalue μ_i if and only if

$$\begin{aligned} 0 &= \|\mathcal{K}\phi_i - \mu_i \phi_i\|^2 \\ &= \langle \mathcal{K}\phi_i - \mu_i\phi_i, \mathcal{K}\phi_i - \mu_i\phi_i \rangle \\ &= \|\mathcal{K}\phi_i\|^2 - \mu_i \langle \mathcal{K}\phi_i, \phi_i \rangle - \mu_i \langle \phi_i, \mathcal{K}\phi_i \rangle + \mu_i^2 \|\phi_i\|^2. \end{aligned} \tag{12}$$

We now perform the Fourier transform on (12). We concentrate on the first two terms since the third term is the complex conjugate of the second and the fourth one translates to $\|\hat{\phi}_i\|^2 = \|\phi_i\|^2$ since the Fourier transform is an L^2 isometry.

Note that by Parseval's identity we have, as in Proposition 4.1,

$$\begin{aligned}\int_{\mathbf{R}} \hat{k}_1(\nu_1, y)\phi_i(y)\, dy &= \int_{\mathbf{R}} \hat{k}(\nu_1,\nu_2)\overline{\widehat{\overline{\phi_i}}(\nu_2)}\, d\nu_2 \\ &= \int_{\mathbf{R}} \hat{k}(\nu_1,\nu_2)\hat{\phi}_i(-\nu_2)\, d\nu_2 \\ &= \int_{\mathbf{R}} \hat{k}(\nu_1,-\nu_2)\hat{\phi}_i(\nu_2)\, d\nu_2 \\ &= \int_{\mathbf{R}} \tilde{k}(\nu_1,\nu_2)\hat{\phi}_i(\nu_2)\, d\nu_2. \end{aligned} \tag{13}$$

The first term in (12) is therefore, using Parseval's identity,

$$\begin{aligned}\|\mathcal{K}\phi_i\|^2 &= \int_{\mathbf{R}}\int_{\mathbf{R}} k(x,s)\phi_i(s) ds\, \overline{\int_{\mathbf{R}} k(x,t)\phi_i(t)\, dt}\, dx \\ &= \int_{\mathbf{R}}\int_{\mathbf{R}} k_1\widehat{(\nu_1,s)}\phi_i(s) ds\, \overline{\int_{\mathbf{R}} k_1\widehat{(\nu_1,t)}\phi_i(t)\, dt}\, d\nu_1 \\ &= \int_{\mathbf{R}}\int_{\mathbf{R}} \hat{k}_1(\nu_1,s)\phi_i(s)\, ds\, \overline{\int_{\mathbf{R}} \hat{k}_1(\nu_1,t)\phi_i(t)\, dt}\, d\nu_1 \\ &= \int_{\mathbf{R}}\int_{\mathbf{R}} \tilde{k}(\nu_1,\sigma)\hat{\phi}_i(\sigma)\, d\sigma\, \overline{\int_{\mathbf{R}} \tilde{k}(\nu_1,\tau)\hat{\phi}_i(\tau)\, d\tau}\, d\nu_1 \\ &= \|\tilde{\mathcal{K}}\hat{\phi}_i\|^2. \end{aligned}$$

Analogously, the second term satisfies

$$\begin{aligned}\langle \mathcal{K}\phi_i, \phi_i\rangle &= \int_{\mathbf{R}} \left[\int_{\mathbf{R}} k(x,y)\phi_i(y)\, dy\right] \overline{\phi_i(x)}\, dx \\ &= \int_{\mathbf{R}} \left[\int_{\mathbf{R}} \tilde{k}(\nu_1,\nu_2)\hat{\phi}_i(\nu_2)\, d\nu_2\right] \overline{\hat{\phi}_i(\nu_1)}\, d\nu_1 \\ &= \langle \tilde{\mathcal{K}}\hat{\phi}_i, \hat{\phi}_i\rangle. \end{aligned}$$

From (12) we conclude that

$$\begin{aligned} 0 &= \|\mathcal{K}\phi_i\|^2 - \mu_i\langle\mathcal{K}\phi_i,\phi_i\rangle - \mu_i\langle\phi_i,\mathcal{K}\phi_i\rangle + \mu_i^2\|\phi_i\|^2 \\ &= \|\tilde{\mathcal{K}}\hat{\phi}_i\|^2 - \mu_i\langle\tilde{\mathcal{K}}\hat{\phi}_i,\hat{\phi}_i\rangle - \mu_i\langle\hat{\phi}_i,\tilde{\mathcal{K}}\hat{\phi}_i\rangle + \mu_i^2\|\hat{\phi}_i\|^2 \\ &= \|\tilde{\mathcal{K}}\hat{\phi}_i - \mu_i\hat{\phi}_i\|^2. \end{aligned}$$

Therefore $\hat{\phi}_i$ is an eigenfunction of $\tilde{\mathcal{K}}$ with eigenvalue μ_i. Reversing the roles of $\mathcal{K}$ and $\tilde{\mathcal{K}}$ we find that this relationship is an equivalence.

To prove (iii), note that since $\|\tilde{k}\|_{L^2} = \|k\|_{L^2}$ by Parseval's identity, it follows that $\|\tilde{\mathcal{K}}\| = \|\mathcal{K}\|$. $\tilde{\mathcal{K}}$ is therefore a Hilbert-Schmidt operator and thus compact. As in Corollary 3.2, it follows from standard operator theory that $\tilde{k}$ is expressed by a bilinear series

$$\tilde{k}(\nu_1, \nu_2) = \sum_{i \geq 0} \mu_i \, \varphi_i(\nu_1) \, \overline{\varphi_i(\nu_2)},$$

where the φ_i are L^2-orthonormal eigenfunctions, the series is L^2 convergent and the $\{\varphi_i\}_{i \geq 0}$ associated to nonzero eigenvalues form a complete orthonormal basis of the range of $\tilde{\mathcal{K}}$. However, it follows from (ii) that if

$$k(x, y) = \sum_{i \geq 0} \mu_i \, \phi_i(x) \, \overline{\phi_i(y)},$$

then

$$\tilde{k}(\nu_1, \nu_2) = \sum_{i \geq 0} \mu_i \, \hat{\phi}_i(\nu_1) \, \overline{\hat{\phi}_i(\nu_2)}$$

has all the required properties, orthonormality of the $\hat{\phi}_i$ and completeness on the range being ensured by Parseval's identity. □

We are now ready to state our main result.

Theorem 4.3. *Suppose k is a continuous L^2-*PDK *such that $k^{1/2}(x, x) \in L^1(\mathbf{R})$ and $k(x, x) \to 0$ as $|x| \to \infty$. Then:*

(i) *$\tilde{k}$ is a continuous L^2-PDK with $\tilde{k}(\nu, \nu) \in L^1(\mathbf{R})$ admitting the bilinear eigenfunction expansion*

$$\tilde{k}(\nu_1, \nu_2) = \sum_{i \geq 0} \mu_i \, \hat{\phi}_i(\nu_1) \, \overline{\hat{\phi}_i(\nu_2)}, \tag{14}$$

which is absolutely and uniformly convergent, where $\{\mu_i\}_{i \geq 0}$ are the eigenvalues of the integral operator $\mathcal{K}$, each $\hat{\phi}_i$ is the Fourier transform of the eigenfunction ϕ_i associated with μ_i, and the $\{\hat{\phi}_i\}_{i \geq 0}$ are uniformly continuous, L^2-orthonormal eigenfunctions of $\tilde{\mathcal{K}}$.

(ii) *The operator* $\tilde{\mathcal{K}}$ *is trace class and*

$$\operatorname{tr}\tilde{\mathcal{K}} = \operatorname{tr}\mathcal{K} = \int_{-\infty}^{+\infty} k(x,x)\,dx = \int_{-\infty}^{+\infty} \tilde{k}(\nu,\nu)\,d\nu = \sum_{i\geq 0}\mu_i.$$

Proof. By Theorem 3.4 $k(x,y)$ admits the expansion

$$k(x,y) = \sum_{i\geq 0}\mu_i\,\phi_i(x)\,\overline{\phi_i(y)}, \tag{15}$$

where the $\{\phi_i\}_{i\geq 0}$ are the eigenfunctions of $\mathcal{K}$, which are uniformly continuous and L^2-orthonormal, μ_i are the eigenvalues of $\mathcal{K}$ and the bilinear series is absolutely and uniformly convergent.

Suppose ϕ_i is a normalized eigenfunction of $\mathcal{K}$ associated with a nonzero eigenvalue μ_i. By Proposition 4.2, $\hat{\phi}_i$ is a normalized eigenfunction of $\hat{\mathcal{K}}$ associated with μ_i. Since by Lemma 3.7 $\phi_i \in L^1(\mathbf{R})$, it follows from the basic properties of the Fourier transform that $\hat{\phi}_i$ is uniformly continuous and $\hat{\phi}_i(\nu) \to 0$ as $|\nu| \to \infty$.

By Proposition 4.2, it follows that

$$\tilde{k}(\nu_1,\nu_2) = \sum_{i\geq 0}\mu_i\,\hat{\phi}_i(\nu_1)\,\overline{\hat{\phi}_i(\nu_2)} \tag{16}$$

holds in L^2. Since $k(x,x) \in L^1(\mathbf{R})$, it follows from Theorem 3.4 that convergence of the series in (15) is uniform, and so applying the Fourier transform we conclude that (16) holds pointwise. Moreover, since from Lemma 3.7 $k \in L^1(\mathbf{R}^2)$ it follows that $\hat{k}$ is uniformly continuous and $\hat{k}(\nu_1,\nu_2) \to 0$ as $|(\nu_1,\nu_2)| \to \infty$.

However, stronger convergence properties are valid. Taking $\nu_1 = \nu_2$ in (16), it follows that

$$S_n(\nu) = \sum_{i=0}^{n}\mu_i\,|\hat{\phi}_i(\nu)|^2 \leq \hat{k}(\nu,\nu)$$

for all n. Thus the sequence of continuous functions $S_n(\nu)$ is monotone increasing and converges pointwise to $\hat{k}(\nu,\nu)$. Since $\hat{k}(\nu,\nu) \to 0$

as $|\nu| \to \infty$, one may apply Dini's theorem to the one-point compactification of $\mathbf{R}$, concluding that convergence is uniform.

Uniform convergence of the series makes it possible to integrate termwise along the diagonal $\nu_1 = \nu_2$, yielding the (possibly divergent) equality

$$\int_{-\infty}^{+\infty} \tilde{k}(\nu,\nu)\,d\nu = \sum_{i\ge 0} \mu_i. \tag{17}$$

However, since $k^{1/2}(x,x) \in L^1(\mathbf{R})$ and $k(x,x) \to 0$ as $|x| \to \infty$, it follows that $k(x,x) \in L^1(\mathbf{R})$ and $k(x,x) \to 0$ as $|x| \to \infty$. By Theorem 3.4 the series on the righthand side of (17) is the trace of $\mathcal{K}$, $\int_{-\infty}^{+\infty} k(x,x)\,dx$. In particular it is, of course, convergent. Thus $\tilde{k}(\nu,\nu) \in L^1(\mathbf{R})$, and the associated operator $\tilde{\mathcal{K}}$ is trace class with trace given by (17).

It then follows from Theorem 3.4 that convergence of the series (16) is absolute and uniform. $\square$

Remark 4.4. As the proof of Theorem 4.3 shows, the condition $k^{1/2}(x,x) \in L^1(\mathbf{R})$ may be replaced by the weaker $k(x,x) \in L^1(\mathbf{R})$ and $k(x,y) \in L^1(\mathbf{R}^2)$, under which Theorem 4.3 is still valid. The version presented, although somewhat weaker, underlines how the behavior of the kernel on the diagonal controls events.

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