COMPARISON OF SMALL SATELLITE ATTITUDE DETERMINATION METHODS

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ABSTRACT

Satellite attitude determination methods usually fall in one of two classes: point-by-point and recursive estimation algorithms. Point-by-point attitude determination is based on the measurements of two or more sensors in a single point in time, while recursive estimation uses information from successive time points, as well as knowledge about the spacecraft dynamics and/or kinematics models. In small satellites, a single attitude sensor is often available, due to cost and space constraints, thus leading to the exploration of recursive estimation based solutions, such as the Kalman filter. In this paper, the results of using a point-by-point Singular Value Decomposition (SVD) algorithm are compared to those obtained by an Extended Kalman Filter (EKF), when applied to a simulation of the small satellite PoSAT-1, which includes on board magnetometers and a Sun sensor. Questions of both theoretical and practical nature are discussed and analysed.

Keywords: Attitude estimation, Quaternions, Small Satellites, Extended Kalman Filter, Wahba Problem.

INTRODUCTION

The determination of the three-axis spacecraft attitude has a major role in guidance, navigation and control of an aerospace vehicle, specially for autonomous systems which are less fault tolerant for environment anomalies than ground-based systems. Currently, supporting the control and navigation in ground stations, specially for small inexpensive spacecrafts, is not feasible. Therefore, algorithms must be found to carry out the task of determining the attitude from a sequence of noisy vector observations typically obtained from inboard sensors, such as star sensors, Sun sensors, Earth sensors, Global Position System (GPS) or magnetometers. Since small satellites are typically in Low Earth Orbit (LEO) and their size plus target cost prevent the use of more powerful actuators, such as inertia wheels, the preferred attitude actuators are those which generate a magnetic momentum that interacts with the Earth's geomagnetic field. These actuators are cheap to build and have long service life since no fuel is used except for electric power which is obtained from the solar panels. Also gas jets are becoming an option for this class of satellites.

This work is part of a Portuguese funded project named ConSat[1] aiming at the study of the dynamics of bodies under the influence of gravitational, aerodynamic and control torques in particular the case of small satellites. The work carried out is included in a project task whose goal is the development of new approaches to the attitude control problem, where attitude determination is essential.

The small satellite PoSAT-1 [25] was used as a case study for the ConSat simulator *SimSat*, developed within the project and described in [20]. Written under Matlab/Simulink, SimSat includes a realistic simulation of PoSAT-1 sensors and actuators as well as of the earth geomagnetic field.

The work presented in this article analyses different approaches to attitude determination and its intrinsic problems. Two distinct methods, a recursive Extended Kalman Filter (EKF) and a point-by-point Singular Value Decomposition (SVD) algorithm are applied to determine PoSAT-1's attitude from magnetometer and Sun sensor readings, using SimSat. Moreover, a sensor fusion approach is addressed by combining those two sensor readings to improve the EKF's efficiency. In the next section, a historical perspective of the main approaches to the small spacecraft attitude determination problem is presented. Simulation Setup Section describes the attitude determination methods tested in this work. Finally, the simulation results of both the EKF and SVD methods

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are compared and conclusions are drawn.

HISTORICAL PERSPECTIVE

The main goal of the attitude determination is to compute the attitude of a body fixed coordinate system (CS) relative to a reference CS, as well as the angular velocity, based in noisy vector measurements taken in both systems. In this work the CS fixed to the body is defined as the Control CS: a right orthogonal coordinate system whose origin coincides with the mass center of the satellite and coincident with its moment of inertia directions. The reference CS is the Orbital CS: a right orthogonal coordinate system whose origin coincides with the mass center of the satellite, with the z axis pointing at the zenith, and the x axis tangent to the orbit plane.

There are two widely applied methodologies to the attitude determination problem: the Kalman Filter and the point-by-point methods. Although both emerged in the sixties, only recently point-by-point methods have been subject to intensive research. In contrast, the Kalman filter has been applied successfully to many different estimation problems since it is well-suited to estimate state vectors of multi-input/output systems based in multi-sensor information.

ITERATIVE METHODS

Kalman filter's adequacy for real-time estimation is due to its recursive processing of noisy sensor measurements to determine the successive minimum variance state estimates. Assuming statistics of the system and measurements noise as known, and taking advantage of the system model, the filter propagates the estimated state from one time point to the next. Nevertheless, there are several problems associated with the application of the Kalman filter to small satellites. The attitude representation is a problematic issue addressed by many authors and reviewed by Shuster [22]. Quaternion representation is the most commonly used satellite attitude estimation because it is not singular for any rotation. However it is subjected to the constraint $qq^T = 1$ in order to maintain orthogonality in the estimated attitude. This constraint as to be taken into account during the implementation of the EKF, otherwise the covariance matrix becomes singular. Leffers *et al.* [10] described three different approaches to circumvent the covariance matrix singularity caused by the quaternion attitude representation. However the angular velocities are obtained from gyroscopes, which are often not used in small spacecrafts because not only they are generally expensive but also they tend to fail. Also to overcome the quaternion problem, Bar-Itzhack *et al.* [2] showed that normalization improves filter convergence and accuracy. Applying the covariance modifications described by Lefferts *et al.* [10], Psiaki and Martel [16], using only magnetometer data, estimated the disturbance torques and also the attitude, angular velocity and the vector part of the quaternion, $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$. The scalar part of the quaternion q_4 is obtained from the estimated vector part using the quaternion constraint. However, it only offers coarse attitude information due to the low accuracy of the observations and the inaccuracy of the knowledge of the Earth's magnetic field, as stressed by the author.

Having a non-linear system and nonlinear measurements, the application of the Kalman filter is only possible after linearization. This may lead to divergence problems in the error covariance matrix. Some solutions were presented by Brown and Hwang [8] in order to handle this problem. Also Bak [1] modelled some of the discrepancies between the process model and the actual behaviour of the spacecraft and gathered them in the covariance error matrix in an EKF using magnetic field measurements. Moreover, Vathsal [23] expanded the process and measurement models to second order. However, this approach increases the complexity of the filter and it is a computational burden. Usually, it is only worthwhile to go for second or higher order techniques in case of extreme system nonlinearities.

The process and measurement noise are assumed to be modelled by a zero-mean Gaussian stochastic process with known covariance, but the covariance matrices must be manually tuned, not necessarily to achieve optimal filter designs but sometimes to increase closed loop attitude control performance. Mook and Junkins [17] developed a new approach, designated as the Minimum Model Error Estimation (MME) method, where the model error is determined during the estimation process. Crassidis and Markley [4] used this approach to estimate the attitude of a real spacecraft without the utilization of gyro measurements. However, the MME filter is a batch estimator. Inspired in his paper, and also based on a predictive tracking scheme introduced by Lu [12], Crassidis and Markley [5] have proposed a predictive filter where the model error is estimated as part of the solution. The filter may take any form, even nonlinear, so the filter is free from covariance propagation, decreasing the computational burden associated with it. Moreover, it can be implemented on-line to filter noisy measurements, estimating quaternion attitude representation and rate trajectories.

The Kalman filter implemented in the SimSat simulator is based in the approach of Psiaki and Martel [16] but uses Sun sensors to improve filter accuracy. To avoid the covariance singularity the filter estimates the vector part of the quaternion reducing the rank of the covariance error matrix. The filter covariance matrices for the process and measurement noises are obtained using genetic algorithm based search [3].

POINT-BY-POINT METHODS

A different approach to the attitude estimation problem consists on determining the attitude based on a sequence of noisy vector measurements. Given a set of $n \geq 2$ vector measurements $b_{1,...,}b_n$ in the body system, and a set of reference vectors $r_{1,...,}r_n$ in the orbit system, there is an orthogonal matrix A (the attitude matrix or direction-cosine matrix) that transforms rotational vectors from the orbital to the body coordinates. The problem of finding the best estimate of the A matrix was posed by Grace Wahba [24] who was the first to choose a least square criterion to define the best estimate, i.e., to find the orthogonal matrix A with determinant 1 that minimizes the loss function

$$L(A) = \frac{1}{2}\sum_{i=1}^{n} w_i \left|b_i - Ar_i\right|^2 \qquad (1)$$

where w_i is a set of positive weights assigned to each measurement and $|.|$ denotes the Euclidean norm. It was proved that the loss function can be rewritten as,

$$L(A) = \lambda_0 - tr(AB)^T \qquad (2)$$

with

$$\lambda_0 = \sum_{i=1}^{n} w_i \text{ and } B = \sum_{i=1}^{n} w_i b_i r_i^T \qquad (3)$$

The loss function will be minimum when the trace of the matrix product AB^T is maximum, under the orthonormality constraint on A.

The q method, introduced by Davenport [6], provides a quaternion-based solution for the Wahba problem, where the attitude quaternion q which minimizes the loss function is the eigenvector of a matrix K, corresponding to K's largest eigenvalue, $\lambda_{\max}$. Shuster [18] presented an implementation of the q method, the QUartenion ESTimator (QUEST) where the purpose is to determine $\lambda_{\max}$ and the corresponding q from the vector observations, which avoids solving the eigenvalue problem explicitly. The main disadvantage of this method is that the measurements are combined to provide an attitude estimate but the combination is not optimal in any statistical sense.

The Singular Value Decomposition (SVD) method, which computes the attitude matrix directly, is very simple and one of the most robust estimators minimizing Wahba's loss function together with q method. However, the q method is faster than the SVD when three or more measurements are available [14]. The Fast Optimal Attitude Matrix (FOAM), introduced by Markely [15], is a variation of the q method which avoids the need to compute the eigenvectors and is faster than the q method, though equally robust.

Shuster also derived a simple expression for the covariance matrix of the Three Axis Attitude Determination (TRIAD) algorithm deduced by Lerner [11] in which, despite the simplicity of the attitude determination, the calculation of the covariance matrix was rather complicated because of the need to compute numerous partial derivatives as differences. In spite of the popularity of the TRIAD algorithm[2] it only can be solved for two observations. This represents a big disadvantage when more measurements are available, since some accuracy is lost. Actually, it is possible to combine the attitude solutions of the various observation pairs. However, this solution tends to be too costly.

On the other hand the deterministic methods that use the vector measurements to obtain the attitude at a given time point require at least two vectors (except the TRIAD) to determine the attitude and it requires weighting of the entire vector measurement. So, all deterministic methods fail when only one set of vector measurements is available (e.g. magnetometer data only), which happens when a solar eclipse occurs. Moreover, it is a single time point batch algorithm, where all measurements that are taken at a previous time are ignored. Bar-Itzack [9] presented a recursive routine derived from the QUEST which takes into account all the past measurements and because of that even one measurement is enough to update the attitude. In order to do so, the REQUEST algorithm uses the kinematic equation[3] to propagate the quaternion obtained from the past measurements till the time t_{k+1} and uses it together with the new measurements of the time t_k. In spite of this, it requires exact knowledge of the angular velocity, relying in gyros measurements.

In general, all the deterministic methods or point-by-point methods compute the attitude matrix efficiently and with much less computational load than the EKF, because they do not use information from the dynamic and kinematic models avoiding the modeling errors that arise in EKF. Therefore, they are very attractive to implement in small satellites with short computational resources. Nevertheless, they all require two vector measurements in order to estimate the attitude. Some researchers have used gyroscopes to obtain the angular velocity, but so far gyros are seldom used in small satellites because they are usually ex-

[2]TRIAD was implemented in many missions, for instance the Small Astronomy Satellite (SAS) or the Atmospheric Explorer Missions (AEM).

[3]see the kinematics equation section.

pensive and are often prone to failures, as referred before.

In this work the SVD algorithm was chosen among all point-by-point methods because, together with the q method, it is the most robust method. Also, since only two attitude sensors are used (Sun sensors and magnetometers) the algorithm is as fast as the q method but simpler to implement.

SIMULATION SETUP

A set of simulation tests were set up to compare the attitude determination performance of an EKF and of the SVD point-by-point method. The two algorithms are compared both concerning the computational cost and the attitude determination accuracy. Under the EKF approach, the attitude vector is estimated by minimizing the state estimate error covariance, based on statistical assumptions concerning the uncertainties, together with a set of noisy sensor measurements. In contrast, the SVD algorithm computes the attitude matrix based only in noisy measurements and without making use of the system models, except when the Sun sensor measurements are not available, a situation where the attitude dynamics equation is used to propagate the last attitude estimates before the loss of the Sun sensor readings.

The computation time of the SVD algorithm is very small when compared with the EKF.

The sensors used in the algorithms are among those onboard PoSAT-1. Even though PoSAT-1 has onboard Sun sensors, magnetometers, an Earth horizon sensor (EHS) and a star sensor, only the Sun sensors and the magnetometers are used in the algorithms. The EHS is very sensitive to attitude variations and perturbations greater than 5^o will saturate the filter [?]. Moreover, variations in weather conditions change the measurements as well. Finally, PoSAT-1's star sensor is currently not working.

The simulation setup used for both algorithms is presented in this section, as well as a detailed explanation of how problems where solved in the implementation of the algorithms. We start by recalling the attitude motion equations and the quaternion-based attitude representation used in the algorithms.

SPACECRAFT MODELS

In this section a brief review of the equations of motion, also known as Euler's equations, for a rotational motion of a rigid body in a plane is presented, as well as the attitude parameterization used.

Attitude representation Quaternions were introduced by Hamilton in 1843, and are the most common attitude parameterization used in a satellite attitude determination system due to their inherent non-singularity for any rotation. Nevertheless, one should keep in mind that, under the quaternion representation, four parameters q_1 , q_2 , q_3 , and q_4 are required to represent a three dimensional attitude vector. This leads to the constraint $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$. Many researchers dedicated a part of their work to quaternion application to spacecraft (e.g., Hughes [7] and Wertz [26]).

Considering q a quaternion, it can be represented, by a **scalar** part and a **vector** part ,

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \mathbf{q} \\ q_4 \end{bmatrix}. \quad (4)$$

Quaternion multiplication must be carefully handled, so that the result is another quaternion [26]:

$$p \otimes q = \begin{bmatrix} q_4 & q_3 & -q_2 & q_1 \\ -q_3 & q_4 & q_1 & q_2 \\ q_2 & -q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} p \quad (5)$$

$$= \begin{bmatrix} p_4 & -p_3 & p_2 & p_1 \\ p_3 & p_4 & -p_1 & p_2 \\ -p_2 & p_1 & p_4 & p_3 \\ -p_1 & -p_2 & -p_3 & p_4 \end{bmatrix} q \quad (6)$$

The attitude matrix that expresses orientation between Orbital CS and Control CS, can be expressed in terms of a quaternion [26],

$$A(q) = \begin{bmatrix} q_1^2-q_2^2-q_3^2+q_4^2 & 2(q_1q_2+q_3q_4) & 2(q_1q_3-q_2q_4) \\ 2(q_1q_2-q_3q_4) & -q_1^2+q_2^2-q_3^2+q_4^2 & 2(q_2q_3+q_1q_4) \\ 2(q_1q_3+q_2q_4) & 2(q_2q_3-q_1q_4) & -q_1^2-q_2^2+q_3^2+q_4^2 \end{bmatrix}$$
$$= (q_4^2 - ||q||^2)I_{3x3} + 2qq^T - 2q_4[q\times] \quad (7)$$

where $[\mathbf{q}\times]$ is a skew symmetric matrix that implements algebraically the cross product between two vectors, $a \times b = [a\times]b$,

$$[\mathbf{a}\times] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \quad (8)$$

Dynamic Equations For an earth orbit satellite whose principal directions, corresponding to the three principal moments of inertia, are aligned with the control CS (causing the products of inertia to be zero), the dynamic equation of motion is well known and can be found in text books such as Hughes [7] or Wertz [26]:

$$I\frac{d}{dt}{}^{c}\omega_{ci} = -{}^{c}\omega_{ci} \times I\ {}^{c}\omega_{ci} + {}^{c}N_{gg} + {}^{c}N_{c} + {}^{c}N_{dt} \quad (9)$$

where

- ${}^{c}\omega_{ci}$ is the angular velocity of the control CS *w.r.t.* inertial CS, expressed in the control CS.
- $I = diag\begin{bmatrix} I_{xx} & I_{yy} & I_{zz} \end{bmatrix}$ is the diagonal inertia matrix whose moments of inertia are the principal moments of inertia. PoSAT-1 was simulated in SimSat using the following inertia matrix (see the discussion concerning the inertia matrix in Kalman Filter Section):

$$I = \begin{bmatrix} 119.14 \pm 0.05 & \pm 0.0005 & \pm 0.0005 \\ \pm 0.0005 & 119.06 \pm 0.05 & \pm 0.0005 \\ \pm 0.0005 & \pm 0.0005 & 0.78 \pm 0.05 \end{bmatrix} \text{kg.m}^2 \quad (10)$$

- ${}^{c}N_{dt}$ is the disturbance moment written in control CS due to aerodynamic drag and solar pressure, eccentricity of the orbit and other effects neglected at the PoSAT-1 altitude[4].
- ${}^{c}N_{gg} = 3\omega_0^2\left({}^{c}K_o \times I{}^{c}K_o\right)$ is the gravity moment written in control CS. The orbital angular velocity of PoSAT-1 is $\omega o = 0.0010385$ rad/s. ${}^{c}K_o$ is the unit vector along z-axis of Orbit CS.
- ${}^{c}N_c = {}^{c}m(t) \times {}^{c}B(t)$ is the control moment (expressed in the control CS) and is generated by the cross coupling between the magnetic moment and the geomagnetic field. Earths's magnetic field was simulated using a spherical harmonic IGRF model, Wertz [26]. Results obtained from the simulator by comparison between the real data and simulated can be found in [21].

It would not be reasonable to implement, in the state propagation stage of the EKF, the complete set of equations of motion considering the products of inertia. This would be translated in large onboard computer memory consumption with no equivalent improvement of the state estimation. Due to this and to PoSAT-1's axi-symmetrical geometry, the EKF implementation uses a diagonal inertia matrix with the following values, $I_{xx} = I_{yy} = 119.1$ kg.m^2 and $I_{zz} = 0.784$ kg.m^2. The small discrepancies in the inertia matrix values adds realistic errors to the EKF implementation

For similar reasons, the controller and estimator algorithms use a 4^{th}order IGRF model for the geomagnetic field, while the simulation of the satellite attitude dynamics is based on a 10^{th} order geomagnetic fields model.

Kinematic Equation The kinematic equation of the satellite, relating its angular velocity to the changes of the attitude matrix over the time, is given by

$$\frac{d}{dt}q(t) = \frac{1}{2}\Omega({}^{c}\omega_{co})q \quad (11)$$

where,

$$\Omega({}^{c}\omega_{co}) = \quad (12)$$

$$= \begin{bmatrix} 0 & \omega_{z/co} & -\omega_{y/co} & \omega_{x/co} \\ -\omega_{z/co} & 0 & \omega_{x/co} & \omega_{y/co} \\ \omega_{y/co} & -\omega_{x/co} & 0 & \omega_{z/co} \\ -\omega_{x/co} & -\omega_{y/co} & -\omega_{z/co} & 0 \end{bmatrix}$$

The angular rates components used are orbital referenced, because the kinematic equations describe the rotation between the orbital axes and the satellite axes. As such, one must relate the orbital and the inertial references:

$${}^{c}\omega_{ci} = {}^{c}\omega_{co} + {}^{c}\omega_{oi} = {}^{c}\omega_{co} + A({}^{c}_{o}q){}^{c}\omega_{oi} =$$

$$= \omega_{co} + \omega o{}^{c}i_o \quad (13)$$

where ${}^{c}A({}^{c}_{o}q) = \begin{bmatrix} {}^{c}i_o & {}^{c}j_o & {}^{c}k_o \end{bmatrix}$ is the unit vector along x-axis of Orbit CS.

Since PoSAT-1 has an orbital inclination of 98º and with an eccentricity of 0.001, the angular velocity of the control CS *w.r.t.* the Inertial CS is approximately given by the following expression, as described by Tavares *et. al.* [21].

$${}^{c}\omega_{oi} = \begin{bmatrix} \omega o & 0 & 0 \end{bmatrix}^T$$

KALMAN FILTER

The Kalman filter guarantees minimum variance state estimation when applied to linear dynamic systems. Nevertheless, the Kalman filter can be used in non-linear problems by linearization of the equations that describe the system[5]. However, this is an approximation that is going to introduce errors in the estimate calculation. Also, another approximation is to assume Gaussian noise system for observations and process with known variance which is not an accurate description of the estimation errors. In addition to this, the products of inertia are all zero because principal directions

[4] Wertz [26].

[5] The equations implemented are described in the Appendix A.

are assumed to be aligned with control CS due to the spacecraft geometry [26]. Also there is some uncertainty in the calibration of the principal moments of inertia. In practice these are systematic errors that cannot be modelled using Gaussian white noise, deviating the estimate from its true parameters. To avoid these problems, the error and process covariance error are manually tuned with the filter in the attitude closed control loop, to compensate the resulting errors. Initially, the covariance error matrix was obtained in open loop comparing the system state and output of a 10^{th} order and 4^{th} geomagnetic field model. The obtained estimation pointing accuracy was of approximately 5^o. Then, instead of manual tuning, which is very time consuming and prone to subjective error, a genetic algorithm was used to tune both the process covariance matrix the error covariance matrix , with the EKF inside the closed control loop, and evaluating performance by measuring the accuracy of the resulting attitude control loop [20]. By exploring the simmetries of the covariance error matrix and of the PoSAT-1 geometry, a reduced group of parameters had to be tuned. Since the measurement errors in PoSAT-1 are only due to geomagnetic field changes, the covariance error is a diagonal matrix with the 0.1 value in the diagonal.

The system modeling of a spacecraft is a complex issue because the dynamic equations of motion take into account the external torques (due to solar pressure, solar heating, aerodynamic drag, eccentricity of the orbit and several other effects). These are non-linear effects and vary along the orbit and altitude, which strongly influence the accuracy of the attitude determination. In practice they are disregarded due to its modelling complexity. This turns out to be a trade-off improvement and computational burden in considering the disturbance moment.

Based in one of the three approaches done by Lefferts *et. al.* [10] to avoid the matrix singularity, the scalar part of the quaternion q_4 is not estimated reducing the rank in one of the matrix involved in the algorithm, the transition matrix Φ, the error covariance matrix P and the covariance of the process Q, the Kalman gain K as well as the F matrix that contains the equations of motion and the H matrix that relates sensor measurements with the state. To reconstruct the full quaternion the fourth element of the quaternion is obtained from the estimated vector part and using the constraint $||q||^2 = 1$ leading to $q_4 = \sqrt{1 - q_1^2 - q_2^2 - q_3^2}$.

When propagating the state and covariance matrices and also to propagate the attitude quaternion, the full quaternion must be handled carefully to obtain a proper rotation. Therefore, in these steps of the algorithm, the quaternions have to be handled separately from the angular velocity. Instead of using $\frac{d}{dt}x_{k+1} = \int f(x(t), u(t), t)dt + x_k^+$, as for the angular velocity, the quaternion must be propagated through the transition matrix $\Phi = e^{\int \frac{1}{2}\Omega(w)dt}$ without approximation, resulting in $q_{k+1}^+ = \left(\cos\left(\frac{\Delta T\omega}{2}\right) + \frac{1}{\omega}\sin\left(\frac{\Delta T\omega}{2}\right)\Omega\left(\omega_k^+\right)\right) q_k^-$ [26]. When updating the state, where the estimated state is $\widehat{x} = \left[\begin{array}{cc} \widehat{\omega}^T & \widehat{\mathbf{q}}^T \end{array}\right]$ the weighted peturbation error, $\Delta\widehat{x}_{k+1} = \left[\begin{array}{cc} \Delta\widehat{\omega}_{k+1} & \Delta\widehat{\mathbf{q}}_{k+1} \end{array}\right]$ estimated by the filter is computed, $\Delta\widehat{x}_{k+1} = K_{k+1}\left(y_{meas,k+1} - A(\widehat{q}_k)y_{orb,k}\right)$ and in case of the angular velocity added to the full state, $\widehat{\omega}_{k+1}^+ = \widehat{\omega}_{k+1}^- + \Delta\widehat{\omega}_{k+1}$. However to preserve physical sense to the quaternion update, the quaternion is updated using quaternion multiplication,

$$\widehat{q}_{k+1}^+ = \left[\begin{array}{c} \frac{\Delta\widehat{q}_{k+1}}{\sqrt{1 - |\Delta\widehat{q}_{k+1}|^2}} \end{array}\right] \otimes \widehat{q}_{k+1}^- \tag{14}$$

When the sun sensor measurements are available the measurement covariance matrix is expanded to a $6X6$ matrix, $R_{tot} = \left[\begin{array}{cc} R_{mag} & 0_{3\times3} \\ 0_{3\times3} & R_{SS} \end{array}\right]$ and the H matrix is expanded in order to incorporate also the Sun sensor measurement, y_{SS}

$$H_{k+1}^+ = \left[\begin{array}{cccc} 0_{3\times3} & \frac{\delta A(q)}{\delta q_1}_{q_1=\widehat{q_1}} y_{orb} & \frac{\delta A(q)}{\delta q_2}_{q_2=\widehat{q_2}} y_{orb} & \frac{\delta A(q)}{\delta q_3}_{q_3=\widehat{q_3}} y_{orb} \\ 0_{3\times3} & \frac{\delta A(q)}{\delta q_1}_{q_1=\widehat{q_1}} y_{SS} & \frac{\delta A(q)}{\delta q_2}_{q_2=\widehat{q_2}} y_{SS} & \frac{\delta A(q)}{\delta q_3}_{q_3=\widehat{q_3}} y_{SS} \end{array}\right] \tag{15}$$

The remaining steps of the algorithm are not modified.

DETERMINISTIC METHODS

One well known approach that disregards the equations of motion, to determine the attitude matrix only from a set of noisy vector measurements, are the deterministic methods or point-by-point methods. The covariance matrix of the process and measurement noises are not used avoiding the problems described previously and the time consumed to calculate and tune the covariances. Moreover a different setting will cause less accurate state estimates in the EFK and the covariance matrix must be retuned.

The deterministic methods take advantage of the sensor measurements only, disregarding the information from the model of the system and not propagating the state estimate. One may think that some useful information from the system model is lost but since the system model is non-linear and has to be linearized it is better to ignore it than to introduce misleading information to the filter. The SVD was chosen because, for two available attitude measurements, it is the most robust (together with the q method) and the fastest point-by-point algorithm [14]. Since the SVD algorithm was implemented in five Matlab code lines, the time taken to estimate the attitude matrix and the covariance of the error is very small compared to the EKF algorithm. The only problem with the SVD algorithm is to obtain the quaternion estimate from attitude matrix. This can be done from Eq. 7.

One of the four solutions is

$$\begin{aligned}\mathbf{q}_4^1 &= \pm 0.5\sqrt{1 + A_{11} + A_{22} + A_{33}}\\ q_1^1 &= 0.25(A_{23} - A_{32})/\mathbf{q}_4^1\\ q_2^1 &= 0.25(A_{31} - A_{13})/\mathbf{q}_4^1\\ q_3^1 &= 0.25(A_{12} - A_{21})/\mathbf{q}_4^1\end{aligned}$$

However numerical inaccuracies may arise when $\mathbf{q}_4$ is very small. One way to overcome this is to compute
the maximum of $\mathbf{q}_4^2 = \pm 0.5\sqrt{1 + A_{11} - A_{22} - A_{33}}$, $\mathbf{q}_4^3 = \pm 0.5\sqrt{1 - A_{11} + A_{22} - A_{33}}$, and $\mathbf{q}_4^4 = \pm 0.5\sqrt{1 - A_{11} - A_{22} + A_{33}}$and based on this, shift among solutions, as suggested by Sidi [19]. The three other solutions are,

$$\begin{aligned}\mathbf{q}_1^2 &= \pm 0.5\sqrt{1 + A_{11} - A_{22} - A_{33}}\\ q_2^2 &= 0.25(A_{12} + A_{21})/\mathbf{q}_1^2\\ q_3^2 &= 0.25(A_{13} + A_{13})/\mathbf{q}_1^2\\ q_4^2 &= 0.25(A_{23} - A_{32})/\mathbf{q}_1^2\end{aligned}$$

$$\begin{aligned}\mathbf{q}_2^3 &= \pm 0.5\sqrt{1 - A_{11} + A_{22} - A_{33}}\\ q_1^3 &= 0.25(A_{12} + A_{21})/\mathbf{q}_2^3\\ q_3^3 &= 0.25(A_{23} + A_{32})/\mathbf{q}_2^3\\ q_4^3 &= 0.25(A_{31} - A_{13})/\mathbf{q}_2^3\end{aligned}$$

$$\begin{aligned}\mathbf{q}_3^4 &= \pm 0.5\sqrt{1 - A_{11} - A_{22} + A_{33}}\\ q_1^4 &= 0.25(A_{13} + A_{31})/\mathbf{q}_3^4\\ q_2^4 &= 0.25(A_{23} + A_{32})/\mathbf{q}_3^4\\ q_4^4 &= 0.25(A_{12} - A_{21})/\mathbf{q}_3^4\end{aligned}$$

SIMULATION RESULTS

To compare the performance of both EKF and SVD algorithms one test consisting of a batch of ten simulations, each one orbit long and with different starting condition was setup as follows:

Test: Each simulation is started at a pitch angle= 60^o and a yaw angle= 0^o. The roll angle is different for each simulation. The initial angular velocity is $\begin{bmatrix} 0.001037 & 0 & 0.02 \end{bmatrix}$rad/s. PoSAT-1 rotates about its longitudinal axis with a spin $w_z = 0.02$ rad/s.

This intends to simulate a situation where the satellite is disturbed by a large initial *w.r.t.* the local vertical.

	γ(degrees)	Spin Rate Error %
mean	0.566	0.167
σ^2	0.595	0.199
worst	1.876	0.696

Table 1: Results for EKF

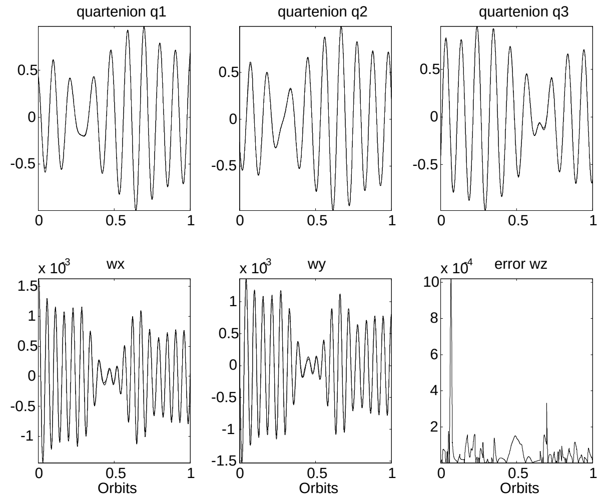

Figure 1: Actual (solid) and EKF estimate (dot) of PoSAT-1 attitude quaternion and angular velocity along one orbit.

Table 1 shows the accuracy of the EKF estimator, determined from the tests. The γ is the angle between the local vertical and the boom axis.

In order to have no influence from a controller the tests are done in open loop with the estimator estimating the full quaternion and the angular velocity.

When the Sun sensor is not available, the state vector is propagated by the dynamics equation, using as initial condition the angular velocity estimate obtained from the inverse kinematics equation, by estimating $\dot{q}$ as the difference between the last two estimated quaternion values before the Sun sensor became available, divided by the sampling time.

As soon as the two sensors (Sun and magnetometer) are available again, the SVD algorithm corrects the quaternion estimate based only on patch orbital location readings. The results for the SVD algorithm are shown in Table 2. As expected, due to the propagation of the quaternion and the angular velocity, the accuracy is not so good as for the EKF, except for the angular velocity. This is evident from the plot in Figure 2, about 0.65 orbits after propagation the state, when the sun sensor was not available. When the two sensor measurements were available again, the SVD estimated the quaternion accurately.

Better results, shown in Figure 1, have been obtained for the EKF by tuning the covariance matrices for a specific orbit (which would result in pointing errors as low as 0.1^o), but these would not work in the general case.

The SVD method does not have similar problem since its estimates are obtained from the sen-

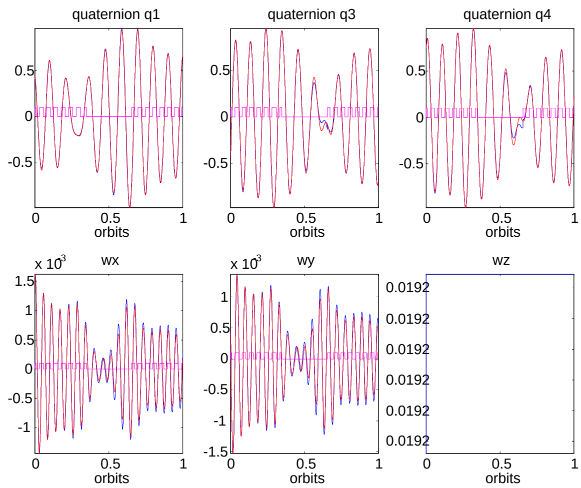

Figure 2: Actual (solid) and SVD estimate (dot) of PoSAT-1 attitude quaternion and angular velocity along one orbit. The plots include a signal which is high when the Sun sensor is available and low otherwise.

sor measurements. Another point is that the SVD does not need initial values of the quaternion as the EKF does. This is a sensitive issue for the EKF because if the initial covariance of the error is too high the filter diverges. Moreover, the computational burden is reduced to half when the SVD algorithm is used.

	γ(degrees)	Spin Rate Error %
mean	0.566	0.167
σ^2	0.595	0.199
worst	1.876	0.696

Table 2: Results for SVD

CONCLUSIONS AND FUTURE WORK

In this paper a point-by-point (SVD) and recursive estimation (EKF) methods for attitude determination were tested on a realistic simulation of the small satellite PoSAT-1, in order to analyse the trade-off between attitude determination accuracy and computational cost. The EKF produces, as expected, the most accurate results, at the cost of increased use of computational resources, due to the computation of the linearized dynamics at each orbital location. The SVD requires the permanent availability of two sensors, but only on magnetometer and one Sun sensor are available onboard PoSAT-1, thus leading to the requirement of propagating the attitude dynamics and kinematics while the Sun sensor is unavailable. This produces poorer results than if SVD had two sensors available along the whole orbit. Future work includes the use of measurements from another sensor always available in space for small satellites (GPS), combined with the magnetometer measurements, to obtain a more fair comparison of the two methods.

Other work will consist of further validating both methods with real data from PoSAT-1, and to include the estimators in the attitude closed control loop. Some work has already been done towards this direction [21].

APPENDIX A
EXTENDED KALMAN FILTER DETERMINATION ALGORITHM

Between measurements

1 Propagation of the state Vector $x = \left[\; {}^c\omega_{x/ci} \;\; {}^c\omega_{y/ci} \;\; {}^c\omega_{z/ci} \;\; {}^c_oq_1 \;\; {}^c_oq_2 \;\; {}^c_oq_3 \;\; {}^c_oq_4 \;\right]^T$
$\frac{d}{dt}\omega^-_{k+1} = \int f(\omega(t), u(t), t)dt + \omega^+_k$ and $q^+_{k+1} = \left(\cos\left(\frac{\Delta T\omega}{2}\right) + \frac{1}{\omega}\sin\left(\frac{\Delta T\omega}{2}\right)\Omega\left(\omega^+_k\right)\right) q^-_k$ where the ΔT is the time between two measurements.

2 Covariance error matrix $P^-_{k+1} = \Phi_k P^+_k \Phi_k + Q_k$ where the transition matrix is $\Phi_k \simeq I + F(t)\Delta T$

Across measurements y_{k+1}

1 Update H matrix $H^+_{k+1} = \left[\; 0_{3\times 3} \quad \frac{\delta A(q)}{\delta q_1}\Big|_{q_1=\widehat{q}_1} y_{orb} \quad \frac{\delta A(q)}{\delta q_2}\Big|_{q_2=\widehat{q}_2} y_{orb} \quad \frac{\delta A(q)}{\delta q_3}\Big|_{q_3=\widehat{q}_3} y_{orb} \;\right]$

2 Compute Kalman Gain $K_{k+1} = P^-_{k+1}H^{-T}_{k+1}[H^-_{k+1}P^-_{k+1}H^-_{k+1} + R_{k+1}]^{-1}$

3 Update estimate $\Delta\widehat{x}_{k+1} = K_{k+1}\left(y_{meas,k+1} - A(\widehat{q}_k)y_{orb,k}\right)$

$$\widehat{\omega}^+_{k+1} = \widehat{\omega}^-_{k+1} + \Delta\widehat{\omega}_{k+1}$$

$$\widehat{q}^+_{k+1} = \left[\begin{array}{c} \Delta\widehat{q}_{k+1} \\ \sqrt{1 - |\Delta\widehat{q}_{k+1}|^2} \end{array}\right] \otimes \widehat{q}^-_{k+1}$$

4 Update H matrix H^+_{k+1} and compute $P^+_{k+1} = [1 - K_{k+1}H^+_{k+1}]P^-_{k+1}[1 - K_{k+1}H^+_{k+1}]^T + K_{k+1}R_{k+1}K^T_{k+1}$

Where $F(t)$ are the linearized equations of motion, $F(t) = \frac{\delta f(x(t),u(t),t)}{\delta x}\Big|_{x=\widehat{x}}$ and used to propagate the error covariance matrix

$$F(t) = \left[\begin{array}{cc} I^{-1}\left([I^c\widehat{\omega}_{ci}\times] - [^c\widehat{\omega}_{ci}\times]\, I\right) & 6\omega_o^2 I^{-1}F_{gg} \\ \frac{1}{2}I_{3x3} & -[\widehat{\omega}_{co}\times] \end{array}\right] \tag{16}$$

where

$$F_{gg} = \begin{bmatrix} \sigma_x\left(\hat{A}_{33}^2-\hat{A}_{23}^2\right) & \sigma_x\left(A_{13}A_{23}\right) & -\sigma_x\left(A_{33}A_{13}\right) \\ -\sigma_y\left(A_{13}A_{23}\right) & \sigma_y\left(A_{33}^2-A_{13}^2\right) & \sigma_y\left(A_{33}A_{23}\right) \\ \sigma_z\left(A_{13}A_{33}\right) & \sigma_z\left(A_{23}A_{33}\right) & \sigma_z\left(A_{23}^2-A_{13}^2\right) \end{bmatrix}$$

where $\sigma_x = \left(I_{zz}^2 - I_{yy}^2\right)$, $\sigma_y = \left(I_{xx}^2 - I_{zz}^2\right)$ and $\sigma_z = \left(I_{yy}^2 - I_{xx}^2\right)$

APPENDIX B SVD ATTITUDE DETERMINATION ALGORITHM

1 Given the measurements y_{k+1} compute $B = \sum_{i=1}^{n} w_i y_{k+1,i} y_{orb,i}{}^T$

2 Compute the singular value decomposition $[U, S, V] = \text{svd}(B)$;

3 Compute the Attitude matrix $A_{opt} = U\ diag \begin{bmatrix} 1 & 1 & det(U)\ det(V) \end{bmatrix} V^T$

4 Compute the covariance error matrix,

$$P = U\ diag\left(\begin{bmatrix} (S_{22}+\sigma)^{-1} & (\sigma+S_{11})^{-1} & (S_{11}+S_{22})^{-1} \end{bmatrix}\right) U^T$$

where $\sigma = det(U)\ det(V) S_{33}$

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