A polar decomposition approach for exact source localization from squared range differences

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***Abstract*—This work considers the problem of locating a single source given a set of squared noisy range difference measurements to a set of points (anchors) whose positions are known. In the sequel, the localization problem is solved in the Least-Squares (LS) sense by writing the source position in polar/spherical coordinates. This representation transforms the original nonconvex/multimodal cost function into the quotient of two quadratic forms, whose maximization is more tractable than the original problem. A solution technique based on the bisection algorithm and Karush-Kuhn Tucker (KKT) conditions is proposed for the resulting optimization problem. Simulation results indicate that the proposed method has similar accuracy to state-of-the-art optimization-based localization algorithms in its class, and the simple algorithmic structure and computational efficiency makes it appealing for applications with strong computational constraints, e.g., in the context of wireless sensor networks.**

***Index Terms*—Squared range difference-based source localization, TDOA, least squares, Karush-Khun Tucker conditions.**

I. Introduction

The problem of locating a source that emits some type of characteristic signal has been widely studied over the past few decades, both for military and industrial applications. In practice, sources can be localized using the time of arrival (TOA), time difference of arrival (TDOA), angle of arrival (AOA), received signal strength (RSS) or combinations of those. In the TOA approach, the source and the sensors need to be synchronized (or operate cooperatively in transponder mode to measure round-trip times), so that a common time reference is available to measure absolute propagation delays. In a wireless rich scattering environment, RSS measurements can be highly volatile and noisy. Angular information is frequently used in several localization applications, but the required directive antennas are costly. Due to these difficulties, this work addresses the TDOA measurement model, for which only the differences of measured arrival times between sensing nodes are required.

Classical TDOA-based self-localization in navigation applications can be realized by intersecting a set of hyperbolas that are the contour lines of constant (measured) range differences between various beacons and the reference. Because of errors in TDOA measurements, these hyperbolas will not intersect at a single point, which leads to mathematically inconsistent localization equations. A common goal is to find an estimate of the source location that minimizes those inconsistencies.

In the literature, there are mainly four approaches to solve the nonlinear system of equations defining the hyperbolic localization problem [1], [2].

The first traditional approach is the reorganization of the nonlinear terms and introduction of additional variables to attain linear equations that can be solved in closed form by LS [3]. While computationally simpler than iterative methods, Maximum Likelihood (ML) or otherwise, the reorganized linear equations are only suitable in practice for sufficiently small measurement noise.

The second approach is based on the nonlinear LS framework where Taylor-series expansion is used for linearization and the solution is obtained iteratively [4]. When TDOA measurements are corrupted by Gaussian noise, the global minimum of the objective function corresponds to the ML location estimate, which has proven asymptotic consistency and efficiency. Although optimum estimation performance can be attained, this method requires a sufficiently precise initialization.

Recently, various Semidefinite Relaxation (SDR) methods, each with its own advantages and drawbacks, were proposed to solve different variations of the hyperbolic localization problem. An approximate ML formulation of TDOA localization is presented in [5], based on an effective relaxation method to transform the original nonconvex optimization problem into a convex one. However, for accurate results all pairwise TDOA measurements between pairs of nodes have to be exhaustively incorporated into a cost function for minimization, which potentially leads to high computational complexity.

An approximate and iterative localization method that can be implemented in a distributed manner is introduced in [6]. It is based on the popular approach of Projection Onto Convex Sets (POCS), modified to accommodate the unbounded hyperbolic sets that arise in TDOA localization. Numerical simulations show that hyperbolic POCS has several desirable features, such as the ability to accurately locate sources outside of the convex hull spanned by the sensors.

In the present work, an exact and globally convergent LS technique denoted *Bisection-KKT* (BKKT) is proposed for the source localization problem using the square of noisy range difference measurements. Although the problem is nonconvex, we recast it into a more tractable form through a novel technique that switches from Cartesian to polar/spherical coordinates. Unlike relaxation-based methods, the actual source

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coordinates are retained as optimization variables, and since other approximations are not required we refer to this approach as *exact*. The method has low computational complexity and several numerical examples show that it is very accurate, even surpassing (albeit slightly) state-of-the-art algorithms based on similar cost functions. Note that algorithms which use plain, rather than squared, range differences might provide better accuracy at the expense of additional computational complexity and/or increased sensitivity to initialization.

This paper is organized as follows. Section II formulates the problem of source localization based on squared TDOAs, discusses solution techniques, and introduces the proposed approach. Simulation results are presented in Section III to evaluate the location estimation performance of the proposed estimator by comparing it with other existing methods and a simple grid search. Finally, conclusions are drawn in Section IV.

Throughout, vectors and matrices are denoted by boldface lowercase and uppercase letters, respectively. The i-th component of a vector $\mathbf{x}$ is written as x_i. The superscript T denotes the transpose of the given real vector or matrix.

II. Problem Formulation

Let $\mathbf{x} \in \mathbb{R}^n$ be the unknown source position, $\mathbf{a}_i \in \mathbb{R}^n$, $i = 1, \ldots, m$ be known sensor positions (anchors) and assume there exists an additional reference sensor (sensor 0) located at the origin. The noiseless range difference between sensor i and the reference is given by

$$d_i = \|\mathbf{x} - \mathbf{a}_i\| - \|\mathbf{x}\|, \quad \text{for} \quad i = 1, \ldots, m. \tag{1}$$

For given d_i, $\mathbf{a}_i$, the set of possible locations $\mathbf{x}$ satisfying (1) is a hyperbola with foci $\mathbf{a}_i$ and $\mathbf{0}$.

In the presence of noise-induced inconsistencies a natural choice is to minimize the sum of residuals between measured range differences and those predicted by a hypothesized source location. However, the source localization problem can also be solved by picking the source location as the minimizer of a so-called equation error, i.e., the minimizer of the difference between functions of the measured range differences and those hypothesized for a given source location [7]. This still leads to exact solutions with the proposed formulation. Specifically, the modified residual is

$$\|\mathbf{x} - \mathbf{a}_i\|^2 - (d_i + \|\mathbf{x}\|)^2 = -2\mathbf{a}_i^T\mathbf{x} - 2d_i\|\mathbf{x}\| - d_i^2 + \|\mathbf{a}_i\|^2$$

yielding the following LS criterion for the source position [8]:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \sum_{i=1}^{m} (-2\mathbf{a}_i^T\mathbf{x} - 2\|\mathbf{x}\|d_i + g_i)^2, \tag{2}$$

where $g_i = \|\mathbf{a}_i\|^2 - d_i^2$. Expanding (2) and dropping constant terms, it can be represented more compactly as

$$\underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{f}^T\|\mathbf{x}\|\mathbf{x} + \mathbf{s}^T\mathbf{x} + e\|\mathbf{x}\| \tag{3}$$

where

$$\mathbf{A} = \sum_{i=1}^{m} (4\mathbf{a}_i\mathbf{a}_i^T + 4d_i^2\mathbf{I}), \quad \mathbf{f} = \sum_{i=1}^{m} 8d_i\mathbf{a}_i, \tag{4}$$

$$e = \sum_{i=1}^{m} -4d_i g_i, \quad \mathbf{s} = \sum_{i=1}^{m} -4\mathbf{a}_i g_i. \tag{5}$$

Expressing the source location in terms of its range and bearing, i.e., $\{\mathbf{x} = r\mathbf{u} : r > 0 \text{ and } \|\mathbf{u}\| = 1\}$, (3) can be written as the following constrained optimization problem:

$$\begin{aligned} \underset{r,\mathbf{u}}{\text{minimize}} \quad & f(r, \mathbf{u}) = r^2(\mathbf{u}^T\mathbf{A}\mathbf{u} + \mathbf{f}^T\mathbf{u}) + r(\mathbf{s}^T\mathbf{u} + e) \\ \text{subject to} \quad & r > 0, \|\mathbf{u}\| = 1. \end{aligned} \tag{6}$$

For a given $\mathbf{u}$, (6) is a quadratic cost function whose unconstrained optimal solution with respect to r is readily found in closed form from the first order stationary condition

$$\nabla_r f(r, \mathbf{u}) = 2r\mathbf{u}^T\mathbf{A}\mathbf{u} + 2r\mathbf{f}^T\mathbf{u} + \mathbf{s}^T\mathbf{u} + e = 0$$

and

$$r^* = -\frac{e + \mathbf{s}^T\mathbf{u}}{2(\mathbf{u}^T\mathbf{A}\mathbf{u} + \mathbf{f}^T\mathbf{u})}. \tag{7}$$

From $\mathbf{A}$ and $\mathbf{f}$ in (4) the denominator of (7) is readily shown to equal to $8\sum_{i=1}^{m}(\mathbf{a}_i^T\mathbf{x} + d_i)^2 > 0$. Moreover, because a solution $\mathbf{x}^*$ to the original problem (2) exists its polar decomposition $\mathbf{x}^* = r^*\mathbf{u}^*$ will solve (6) and make the numerator of (7) negative. Substituting the optimal r into (6) leads to

$$\begin{aligned} p^* = \underset{\mathbf{u}}{\text{maximize}} \quad & \frac{(e + \mathbf{s}^T\mathbf{u})^2}{4(\mathbf{u}^T\mathbf{A}\mathbf{u} + \mathbf{f}^T\mathbf{u})} \\ \text{subject to} \quad & \|\mathbf{u}\| = 1, \quad e + \mathbf{s}^T\mathbf{u} < 0. \end{aligned} \tag{8}$$

Problem (8) can be solved in two ways:

A. Direct Search Method

When $n = 2$ or $n = 3$ a straightforward (and practical) approach to solve (8) is to perform a 1D or 2D grid search for a near-optimal $\mathbf{u}$. For instance, in 2D vector $\mathbf{u}$ is parametrized as $[\cos\theta \quad \sin\theta]^T$, where $\theta \in (0, 2\pi]$ and for 3D, $\mathbf{u} = [\sin\theta\cos\psi \quad \sin\theta\sin\psi \quad \cos\theta]^T$, where $\theta \in (0, \pi]$ and $\psi \in (0, 2\pi]$. It is a simple method and if the search is fine enough, it finds a good approximation to the source location. It is possible to bound the Lipschitz constant of the objective function to be evaluated, from which a sufficiently coarse grid that adequately brackets its maximum can be derived [9]. One may then resort to a zoom-in procedure to attain the desired accuracy.

B. Bisection Method

Before showing the application of the bisection method to (8), we define an interval that is known to contain the optimal value of (8) and where the bisection search is done.

Problem (8) can be equivalently stated as

$$\begin{aligned} \underset{\mathbf{u},\mathbf{v}}{\text{maximize}} \quad & \frac{(e + \mathbf{s}^T\mathbf{u})^2}{4(\mathbf{v}^T\mathbf{A}\mathbf{v} + \mathbf{f}^T\mathbf{v})} \\ \text{subject to} \quad & \|\mathbf{u}\| = 1, \quad e + \mathbf{s}^T\mathbf{u} < 0, \\ & \|\mathbf{v}\| = 1, \quad e + \mathbf{s}^T\mathbf{v} < 0, \\ & \mathbf{u} = \mathbf{v}, \end{aligned} \tag{9}$$

and the last equality constraint is dropped to obtain a relaxed form

$$\begin{aligned} \underset{\mathbf{u},\mathbf{v}}{\text{maximize}} \quad & \frac{(e + \mathbf{s}^T\mathbf{u})^2}{4(\mathbf{v}^T\mathbf{A}\mathbf{v} + \mathbf{f}^T\mathbf{v})} \\ \text{subject to} \quad & \|\mathbf{u}\| = 1, \quad e + \mathbf{s}^T\mathbf{u} < 0, \\ & \|\mathbf{v}\| = 1, \quad e + \mathbf{s}^T\mathbf{v} < 0. \end{aligned} \tag{10}$$

Optimization problem (10) can be expanded into two problems to separately maximize the numerator and minimize the denominator. The ratio of their optimum values gives the upper bound of the interval, $\bar{p}$. For the lower bound, $\underline{p}$, 0 or $-\bar{p}$ might be chosen.

The bisection method checks if $p^* \geq t$ at the midpoint of the interval, $t = (\bar{p} + \underline{p})/2$, and updates the interval at each iteration until its length is below a given threshold. The feasibility problem is

$$\begin{array}{ll} \text{find} & \mathbf{u} \\ \text{subject to} & \frac{(e+\mathbf{s}^T\mathbf{u})^2}{4(\mathbf{u}^T\mathbf{A}\mathbf{u}+\mathbf{f}^T\mathbf{u})} \geq t, \\ & \|\mathbf{u}\| = 1, \quad e + \mathbf{s}^T\mathbf{u} < 0. \end{array} \tag{11}$$

Equivalently,

$$\begin{array}{ll} \text{find} & \mathbf{u} \\ \text{subject to} & \mathbf{u}^T\mathbf{M}\mathbf{u} + 2\mathbf{b}^T\mathbf{u} + \delta \leq 0, \\ & \|\mathbf{u}\| = 1, \quad e + \mathbf{s}^T\mathbf{u} < 0, \end{array} \tag{12}$$

where $\mathbf{M} = 4t\mathbf{A} - \mathbf{s}\mathbf{s}^T$, $2\mathbf{b} = 4t\mathbf{f} - 2e\mathbf{s}$ and $\delta = -e^2$. However, this is equivalent to checking if the optimal value of the following optimization problem is less than 0 or not

$$\begin{array}{ll} \underset{\mathbf{u}}{\text{minimize}} & \mathbf{u}^T\mathbf{M}\mathbf{u} + 2\mathbf{b}^T\mathbf{u} + \delta \\ \text{subject to} & \|\mathbf{u}\| = 1, \quad e + \mathbf{s}^T\mathbf{u} < 0. \end{array} \tag{13}$$

Problem (13) is a variation of the trust region subproblem, for which optimality conditions and approaches to obtain the global minimizer are known [10]. An efficient method will be introduced to exactly solve (13) using KKT conditions in the sequel.

Since $\mathbf{M}$ is a symmetric matrix, it is decomposed as $\mathbf{M} = \mathbf{Q}\mathbf{D}\mathbf{Q}^T$, with diagonal $\mathbf{D}$ and $\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$, to re-express (13) as

$$\begin{array}{ll} \underset{\mathbf{v}}{\text{minimize}} & \mathbf{v}^T\mathbf{D}\mathbf{v} + 2\mathbf{c}^T\mathbf{v} + \delta \\ \text{subject to} & \|\mathbf{v}\| = 1, \quad e + \mathbf{s}^T\mathbf{Q}\mathbf{v} < 0, \end{array} \tag{14}$$

where $\mathbf{v} = \mathbf{Q}^T\mathbf{u}$ and $\mathbf{c} = \mathbf{Q}^T\mathbf{b}$.

For any optimization problem with differentiable objective and constraint functions for which strong duality holds, any set of primal and dual optimal points must satisfy the KKT conditions [11]. The Lagrangian of (14) with dual variables λ and γ is defined as

$$L(\mathbf{v}, \lambda, \gamma) = \mathbf{v}^T\mathbf{D}\mathbf{v} + 2\mathbf{c}^T\mathbf{v} + \delta + \lambda(\mathbf{v}^T\mathbf{v} - 1) + \gamma(e + \mathbf{s}^T\mathbf{Q}\mathbf{v}). \tag{15}$$

The KKT conditions

$$\nabla_{\mathbf{v}} L(\mathbf{v}^*, \lambda^*, \gamma^*) = 0, \tag{16}$$

$$e + \mathbf{s}^T\mathbf{Q}\mathbf{v}^* < 0, \tag{17}$$

$$\mathbf{v}^{*T}\mathbf{v}^* = 1, \tag{18}$$

$$\gamma^* \geq 0, \tag{19}$$

$$\gamma^*(e + \mathbf{s}^T\mathbf{Q}\mathbf{v}^*) = 0 \tag{20}$$

are satisfied by the primal-dual optimal points $(\mathbf{v}^*, \lambda^*, \gamma^*)$. From conditions (17), (19) and (20), it is obvious that $\gamma^* = 0$ and

$$\nabla_{\mathbf{v}} L(\mathbf{v}, \lambda) = (\mathbf{D} + \lambda\mathbf{I})\mathbf{v} + \mathbf{c} = 0, \tag{21}$$

$$\mathbf{v}^T\mathbf{v} = \mathbf{c}^T(\mathbf{D} + \lambda\mathbf{I})^{-T}(\mathbf{D} + \lambda\mathbf{I})^{-1}\mathbf{c} = 1. \tag{22}$$

Since $\mathbf{D} = \text{diag}(\sigma_1 \ldots \sigma_n)$, where σ_i denotes an eigenvalue of $\mathbf{M}$, λ can be found by calculating the roots of the polynomial

$$\sum_{i=1}^{n} \frac{c_i^2}{(\sigma_i + \lambda)^2} = 1.$$

For example for 2D, we have a 4^{th}-degree polynomial

$$\lambda^4 + 2(\sigma_1 + \sigma_2)\lambda^3 + (4\sigma_1\sigma_2 + \sigma_2^2 + \sigma_1^2 - c_1^2 - c_2^2)\lambda^2 +$$
$$2(\sigma_1^2\sigma_2 + \sigma_1\sigma_2^2 - \sigma_2 c_1^2 - \sigma_1 c_2^2)\lambda + \sigma_1^2\sigma_2^2 - c_1^2\sigma_2^2 - c_2^2\sigma_1^2 = 0,$$

whose four roots correspond to four possibilities for critical points $\mathbf{v} = -(\mathbf{D} + \lambda\mathbf{I})^{-1}\mathbf{c}$. Evaluating the objective function of (14) and checking the constraint $e + \mathbf{s}^T\mathbf{Q}\mathbf{v} < 0$ at these points, the global minimum can be found. It is an exact and very fast method. The major requirement is to calculate the roots of a polynomial of degree 4 (for 2D) or 6 (for 3D). This method will be called BKKT.

III. Simulations and Comparisons with Existing Methods

An exact solution to the problem of source localization using squared range differences (SRD-LS) is given in [8], solving a quadratic objective function subject to two quadratic constraints. Another popular approximate solution to (2), the so-called Spherical Interpolation (SI) method, is based on closed-form linear approximation techniques [7]. In the sequel, our proposed method (BKKT) and the simple grid search discussed in Section II-A (SEARCH) are compared with SRD-LS and SI.

Example 1: To investigate the accuracy of the methods, two physical scenarios are set with a source located in the near field and far field of the sensors. The performance metric is root mean square error (RMSE), defined as $\sqrt{\frac{1}{K}\sum_{k=1}^{K}\|\mathbf{x} - \hat{\mathbf{x}}^k\|^2}$, where $\hat{\mathbf{x}}^k$ denotes an estimated source position in the k-th Monte Carlo run. The number of Monte Carlo runs is 1000 for each noise level.

Near Field Case: In this part of the example, the methods will be compared using five anchors plus a reference sensor at the origin. In each Monte Carlo run the anchor locations $\mathbf{a}_i$ and the source $\mathbf{x}$ were randomly generated from a uniform distribution over the square $[-10, 10] \times [-10, 10]$ m. The observed range-difference measurements were obtained by adding a normal random variable with mean zero and standard deviation $\sigma_{\text{gaussian}} \in [10^{-4}, 10^{-1}]$ m to the exact range differences. Figure 1 shows the positions of anchors, hyperbolas defined by each anchor-reference sensor pair and its measured range difference, as well as the real source position and its estimate by BKKT and SRD-LS in one specific Monte Carlo run for the near field case. The true and estimated source position appear at the intersection of the so-called right branches of the hyperbolas. Table I lists the RMSE of the methods. The RMSE of SEARCH, BKKT and SRD-LS are similar with a slight superiority of BKKT and better than the approximate method SI.

Far Field Case: An array with 10 anchors plus one reference sensor is considered. In each run, the coordinates of 10 anchors that are not located at the origin were randomly generated from a uniform distribution over the square

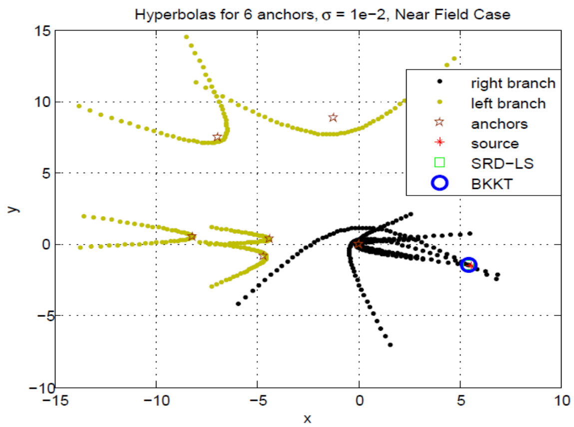

Fig. 1: Hyperbolas and the source position estimated by BKKT and SRD-LS methods.

TABLE I: RMSE comparisons of SEARCH, BKKT, SRD-LS and SI for near field case.

σ_{gaussian}	SEARCH	BKKT	SRD-LS	SI
1e-4	0.0215	0.0190	0.0204	0.0230
1e-3	0.0574	0.0572	0.0568	0.0603
1e-2	0.1646	0.1646	0.1647	0.1808
1e-1	0.5604	0.5604	0.5605	0.6117

$[-10 \quad 10] \times [-10 \quad 10]$ m and the coordinates of the source were randomly generated from a uniform distribution over the square $[-200 \quad -190] \times [-200 \quad -190]$ m. The observed range-difference measurements were obtained as described previously. Table II shows the RMSE of the methods for $\sigma_{\text{gaussian}} \in [10^{-4}, 10^{-1}]$ m. Again, the results of the exact methods are nearly identical and better than the approximate one. The RMSEs are considerably higher than in Table I, as the source is now always located outside of the convex hull of the anchors, and localizing it becomes harder.

Example 2: This example is provided for direct comparison with Example 3 in [8]. Consider an array of $m = 5$ sensors in the plane ($n = 2$) whose coordinates are given by $\mathbf{a}_1 = (-5, -13)$ m, $\mathbf{a}_2 = (-12, 1)$ m, $\mathbf{a}_3 = (-1, -5)$ m, $\mathbf{a}_4 = (-9, -12)$ m, $\mathbf{a}_5 = (-3, -12)$ m. The source coordinates are $\mathbf{x} = (-5, 11)$ m. The observed range differences were obtained by adding white Gaussian noise with standard deviation 0.2 m to the exact range differences. The exact range-differences and their noisy observations are given by

Exact: 11.9170 0.1235 4.4094 11.2622 11.0037

Noisy: 11.8829 0.1803 4.6399 11.2402 10.8183.

SRD-LS has a two-step solution procedure. The first step, which resorts to a bisection method and a root finding technique, has a similar computational complexity to BKKT and both of them have a running time less than a second for the scenarios given in this paper. As it is numerically shown in [8], the first step often fails for problems with high noise levels. When the first step fails, i.e., when the last component of a solution vector is negative, it invokes the second step that satisfies necessary optimality conditions.

$\mathbf{x}_{\text{first step SRD-LS}} = (-7.1645, -12.2497)$,

TABLE II: RMSE comparisons of SEARCH, BKKT, SRD-LS and SI for far field case.

σ_{gaussian}	SEARCH	BKKT	SRD-LS	SI
1e-4	0.3156	0.3222	0.3224	0.3270
1e-3	1.1821	1.1823	1.1834	1.2154
1e-2	3.6724	3.6725	3.6724	3.8401
1e-1	13.303	13.303	13.303	13.397

$\mathbf{x}_{\text{SI}} = (-6.5644, -6.0209)$,
$\mathbf{x}_{\text{SEARCH}} = (-4.9800, 10.2834)$,
$\mathbf{x}_{\text{BKKT}} = (-4.9798, 10.2786)$ and
$\mathbf{x}_{\text{second step SRD-LS}} = (-4.9798, 10.2786)$.

For this setup SRD-LS is unable to give an accurate result without the second step which adds additional computational complexity.

IV. Conclusion

A new algorithm for source localization based on squared range differences was proposed and numerically evaluated. The approach describes the source location in polar coordinates, leading to a non-convex algorithm whose solution can be efficiently found from the KKT conditions through bisection. The algorithm is very precise, yielding localization accuracies that are (slightly) better than those of state-of-the-art algorithms, including others based on root-finding techniques. In addition to high accuracy and computational efficiency, the proposed method has a simple (non-branched) algorithmic structure that may be very appealing in practical implementations.

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