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On minimizing a quadratic function on Stiefel manifold

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ABSTRACT

In this paper we propose a novel approach to a particular quadratic programming problem, when the optimization is performed over the set $\mathsf{O}(3, 2)$ of 3×2 Stiefel matrices. We rewrite the original nonconvex problem as a semi-definite programming problem, by computing a convex hull (tight convex relaxation) of a certain set of matrices. We give an efficient, quick algorithm for the minimization of a quadratic function over Stiefel manifold. We report some numerical experiments to illustrate the tightness of the convex approximation obtained by the two aforementioned methods ("standard" and ours). Our result is of immediate interest in Computer Vision, including Structure-from-Motion (SfM) problems, and 2D–3D registration.

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1. Introduction

By a quadratic programming problem (QP) we understand the following optimization problem:

Given matrices $A_i \in \mathbb{R}^{n\times n}$, $b_i \in \mathbb{R}^{n\times 1}$ and $c_i \in \mathbb{R}$, $i = 0, \ldots, k$, solve

$$\begin{array}{ll} \underset{\substack{\text{subject to}\ \ q^T A_1 q + b_1^T q + c_1 \geq 0 \\ q^T A_2 q + b_2^T q + c_2 \geq 0 \\ \vdots \\ q^T A_k q + b_k^T q + c_k \geq 0}}{\text{minimize}} & q^T A_0 q + b_0^T q + c_0 \end{array} \tag{1}$$

where $q \in \mathbb{R}^{n\times 1}$ is a variable.

Quadratic Programming problems frequently appear in various fields, like control theory [4,5], optimization [1], computer vision, etc. Particular problems that we will be dealing with in this paper have also strong applications in Procrustes' type problems (see Section 4).

The difficulty in solving Quadratic Programming problems (1) depends heavily on the number of constraints involved (k), on whether the cost function is homogeneous or not ($b_0 = 0$), as well as on the properties of the matrices involved. The general theory for solving (efficiently) Quadratic Programming problems, can deal only with some subcases. For a nice exposition of known methods, see [5] and the references therein.

In particular, there exists a solution to problem (1) if there is only one constraint ($k = 1$), or if the cost function is homogeneous and we have two constraints ($b_0 = 0$ and $k = 2$).

In this paper we focus on one particular, but very important, Quadratic Programming problem. Namely, we focus on the case when the variable q runs through vectors of 3×2 Stiefel matrices, i.e., of 3×2 matrices whose columns are orthogonal vectors of unit norm. Also, we shall focus on the homogeneous case.

Thus our main problem is the following one:

Problem 1. For given $C \in \mathbb{R}^{6\times 6}$ solve

$$\underset{\text{subject to}\ \ q = \operatorname{vec}(Q)}{\text{minimize}} \quad q^T C q \tag{2}$$

where Q runs through the Stiefel matrices, i.e.,

$$Q \in \mathsf{O}(3, 2) = \{Q \in \mathbb{R}^{3\times 2} : Q^T Q = I_2\}.$$

As we shall show below, the set of all the constraints in Problem 1 can be characterized by three quadratic constraints. Thus Problem 1 is a QP problem which is beyond the scope of standard techniques.

In this paper, we propose a completely novel approach to Problem 1. In Section 2, we present a very fast algorithm that gives the solution, by using semi-definite programming (SDP) that finds the minimum of linear functions on a convex set of matrices.

In fact, we rewrite Problem 1 as a minimization of a linear function over a certain set of 6×6 matrices. The set over which we are minimizing is non-convex, and we compute its convex hull (tight convex relaxation). We conjecture that it can be computed by introducing one novel constraint. Not only we manage to rewrite the original, non-convex problem, as an equivalent problem, but surprisingly we can describe the convex hull explicitly by using only Linear Matrix Inequalities (only linear functions of the entries of the matrix are involved). Consequently, we managed to rewrite a Problem 1 as a Semi-Definite Programming (SDP) problem [1,2], hence easily solvable via SeDuMi MATLAB toolbox [6].

Our result is of immediate interest for so-called structure from motion (SfM) problems in computer vision. In [3] it proved to be very effective in solving SfM problems for nonrigid (deformable) shapes. In Section 4, we show that it can also be used for Procrustes-like problems on $\mathsf{O}(3,2)$.

In Section 5 we give the results of numerical experiments to illustrate the tightness of the convex approximation obtained by the two aforementioned methods ("standard" and ours), that clearly show the superiority of our method.

2. Computing the convex hull

For a real symmetric matrix $A \in \mathbb{R}^{n\times n}$, we write

$$A \succcurlyeq 0$$

if it is positive semi-definite, i.e., if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$.

For two real symmetric matrices of the same size, $A, B \in \mathbb{R}^{n\times n}$, we write

$$A \succcurlyeq B \quad \text{iff} \quad A - B \succcurlyeq 0.$$

Problem 1 is indeed a quadratic programming problem since the vector $q \in \mathbb{R}^{6\times 1}$ is of the form $\operatorname{vec}(Q)$ for some Stiefel matrix Q if and only if

$$q^T \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} q = 1,$$

$$q^T \begin{bmatrix} 0 & 0 \\ 0 & I_3 \end{bmatrix} q = 1,$$

$$q^T \begin{bmatrix} 0 & I_3 \\ 0 & 0 \end{bmatrix} q = 0.$$

Moreover, since we are minimizing a quadratic function over the set given by three quadratic restrictions, it is beyond the scope of known general techniques (see [4]).

Our problem can be re-written in the following way:

$$\min_{q=\mathrm{vec}(Q)} q^T C q = \min_{q=\mathrm{vec}(Q)} \mathrm{Tr}(Cqq^T) = \min_{X\in S} \mathrm{Tr}(CX).$$

Here S is the set of all matrices of the form qq^T, with $q = \mathrm{vec}(Q)$, for some Stiefel matrix $Q \in \mathbb{R}^{3\times 2}$. The set S can be equivalently described as a set of all real symmetric 6 by 6 matrices

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \tag{3}$$

with $X_{11} \in \mathbb{R}^{3\times 3}$, satisfying the following

$$X \succcurlyeq 0, \tag{4}$$

$$\mathrm{Tr}(X_{11}) = \mathrm{Tr}(X_{22}) = 1, \quad \mathrm{Tr}(X_{12}) = 0, \tag{5}$$

$$\mathrm{rank}\, X = 1. \tag{6}$$

Because of the rank constraint, the set S is non-convex.

Denote by $\mathrm{co}(S)$ the convex hull of the set S, i.e. the set of all convex combinations $c_1X_1 + c_2X_2 + \ldots + c_kX_k$, where $X_i \in S$, $i = 1, \ldots, k$, and c_i's are nonnegative real numbers such that $c_1 + c_2 + \cdots + c_k = 1$. In other words, the convex hull of the set S ($\mathrm{co}(S)$) is the smallest convex set (with respect to inclusion) that contains the set S.

Since the cost function is linear we have

$$\min_{X\in S} \mathrm{Tr}(CX) = \min_{X\in \mathrm{co}(S)} \mathrm{Tr}(CX).$$

The convex hull $\mathrm{co}(S)$ cannot be obtained by simply loosening the rank constraint (6) in the definition of the set S ("standard" convex relaxation), as can be shown by the following example:

Example 1. The matrix

$$M = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

satisfies the conditions (4) and (5). However, it doesn't belong to the convex hull of S. Indeed, if there exist nonnegative numbers $c_1, \dots, c_k$ such that $c_1 + c_2 + \cdots + c_k = 1$, and the matrices $M_1, \dots, M_k \in S$, such that

$$M = c_1 M_1 + \cdots + c_k M_k,$$

then we would have that in matrices $M_1, \dots, M_k$, entries at the positions $(2,2)$, $(3,3)$, $(5,5)$ and $(6,6)$ are zero. Consequently all entries in the second, third, fifth and sixth rows and columns are zero (all matrices are positive semi-definite). However, matrix M_1 being from S is of the form qq^T for some $q = \mathrm{vec}(Q)$, and thus the corresponding matrix Q would be of the form

$$Q = \begin{bmatrix} * & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The last is impossible, since Q is a Stiefel matrix.

Thus, we want to compute $\mathrm{co}(S)$ by introducing some novel constraints instead of the rank constraint (6), and if possible to describe $\mathrm{co}(S)$ by linear matrix inequalities.

Let $Q \in \mathbb{R}^{3\times 2}$ be a Stiefel matrix, and denote its columns by q_1 and q_2. Then the vector $q = \mathrm{vec}(Q) \in \mathbb{R}^{6\times 1}$ is given by $q^T = [q_1^T \ q_2^T]$, and the matrix $X = qq^T$ belongs to S.

The vectors q_1, q_2 and their cross-product $q_1 \times q_2$ form an orthonormal basis of $\mathbb{R}^3$, and consequently the sum of projectors to these three vectors is equal to the identity matrix I_3. Moreover, we have access to the entries of $q_1 \times q_2$ as linear functions of the entries of the off-diagonal block X_{12}. So, we have that the matrices $X \in S$ satisfy

$$vv^T + X_{11} + X_{22} = I_3, \tag{7}$$

where

$$v = v(X) := \begin{bmatrix} b_{23} - b_{32} \\ b_{31} - b_{13} \\ b_{12} - b_{21} \end{bmatrix} \tag{8}$$

with $X_{12} = [b_{ij}]$.

Let $Y \in \mathrm{co}(S)$, and let

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix},$$

with $Y_{11} \in \mathbb{R}^{3\times 3}$. Let

$$v' = v(Y) := \begin{bmatrix} b'_{23} - b'_{32} \\ b'_{31} - b'_{13} \\ b'_{12} - b'_{21} \end{bmatrix},$$

where $Y_{12} = \left[b'_{ij}\right]$. Then we have that

$$v'v'^T + Y_{11} + Y_{22} \preccurlyeq I_3. \tag{9}$$

Indeed, as we saw above, all matrices from S satisfy (7) and hence (9). Moreover, if matrices Y' and Y'' satisfy (9) (the corresponding vectors $v(Y')$ and $v(Y'')$ are denoted by v_1 and v_2, respectively), and if c_1 and c_2 are nonnegative real numbers such that $c_1 + c_2 = 1$, then the matrix $Z := c_1Y' + c_2Y''$ also satisfies (9):

$$\begin{aligned} v(Z)v(Z)^T + Z_{11} + Z_{22} &= (c_1v_1 + c_2v_2)(c_1v_1^T + c_2v_2^T) + c_1Y'_{11} + c_2Y''_{11} \\ &\quad + c_1Y'_{22} + c_2Y''_{22} \\ &= c_1(v_1v_1^T + Y'_{11} + Y'_{22}) + c_2(v_2v_2^T + Y''_{11} + Y''_{22}) \\ &\quad - c_1c_2(v_1 - v_2)(v_1 - v_2)^T \\ &\preccurlyeq c_1I_3 + c_2I_3 = I_3. \end{aligned}$$

Furthermore, we can write (9) as a Linear Matrix Inequality

$$\begin{bmatrix} I_3 - Y_{11} - Y_{22} & v' \\ v'^T & 1 \end{bmatrix} \succcurlyeq 0, \tag{10}$$

since all entries of the matrix on the LHS of (10) are linear functions of the entries of the matrix Y. It is straightforward to see that this new condition easily discards the matrix from Example 1.

We define the set Σ of all real, symmetric matrices $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \in \mathbb{R}^{6\times 6}$, $X_{ij} \in \mathbb{R}^{3\times 3}$, that satisfy:

$$X \succcurlyeq 0, \tag{11}$$

$$\mathrm{Tr}(X_{11}) = \mathrm{Tr}(X_{22}) = 1, \tag{12}$$

$$\mathrm{Tr}(X_{12}) = 0, \tag{13}$$

$$\begin{bmatrix} I_3 - X_{11} - X_{22} & v(X) \\ v(X)^T & 1 \end{bmatrix} \succcurlyeq 0. \tag{14}$$

Here the vector $v(X)$ is given as the following linear function of the entries of the off-diagonal block X_{12}:

$$v(X) = \begin{bmatrix} b_{23} - b_{32} \\ b_{31} - b_{13} \\ b_{12} - b_{21} \end{bmatrix} \tag{15}$$

with $X_{12} = [b_{ij}]$.

We have proved the following:

Theorem 2. *For the above defined sets S and Σ, we have*

$$\mathrm{co}(S) \subset \Sigma$$

Moreover, we conjecture that the converse is also valid:

Conjecture 3.

$$\mathrm{co}(S) = \Sigma.$$

Although we don't have the complete rigorous proof of Conjecture 3, we have strong evidence of its validity. First of all, we have run the tests on very large number of randomly generated matrices ($\geq$ 1000), and the results were always correct, i.e. each randomly generated matrix from Σ was always in the convex hull of the set S (for details see Section 5). Also, we have rigorous proofs for some particular cases – for the complete proof of Conjecture 3 in these particular cases see Section 3. As can be seen from these proofs, they are quite involved and technical, and we expect that the general proof will be along the same lines.

3. Proof of the two particular cases

In order to prove Conjecture 3, we are left with proving

$$\Sigma \subset \mathrm{co}(S).$$

Hence, we need to prove that every matrix $X \in \Sigma$ can be written as a convex combination of the matrices from S, i.e. that there exist nonnegative real numbers $c_1, \dots, c_k \geq 0$, with $\sum_{i=1}^k c_i = 1$, and matrices $X_1, \dots, X_k \in S$, such that

$$X = \sum_{i=1}^{k} c_i X_i.$$

So, let $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ be an arbitrary matrix in Σ. First of all, note that if $P \in \mathbb{R}^{3\times 3}$ is an orthogonal matrix, then the matrix X is from Σ (respectively, from S) if and only if the matrix

$$\begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} P^T & 0 \\ 0 & P^T \end{bmatrix}, \tag{16}$$

is from Σ (respectively, from S). Hence, it is enough to show that for some orthogonal matrix P, the matrix (16) is from $\mathrm{co}(S)$.

Denote by

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

the vector $v(X)$ given by (15). Then the condition (14) in the definition of the set Σ can be written as

$$X_{11} + X_{22} + vv^T \preccurlyeq I_3,$$

and since X_{11} and X_{22} are positive semi-definite, we have that $||v|| \le 1$.

In the following two subsections we give a complete proof of Conjecture 3 in the cases $||v|| = 1$, and $\mathrm{rank}(X_{11} + X_{22}) = 2$, respectively.

3.1. Case $||v|| = 1$

Let P be a matrix from $SO(3)$ such that $Pv = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Since for every orthogonal matrix $Q \in \mathbb{R}^{3\times 3}$ we have that

$$Qv(X) = v\left(\mathrm{diag}(Q, Q) X \,\mathrm{diag}(Q^T, Q^T)\right) \det(Q),$$

we obtain that $v\left(\mathrm{diag}(P, P) X \,\mathrm{diag}(P^T, P^T)\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Therefore, by using the transformation (16) for this matrix P, without loss of generality, we can assume that X is such that $v = v(X) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and so by the definition of $v(X)$:

$$X_{12} - X_{12}^T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{17}$$

Thus, we are left with proving that all matrices $X \in \Sigma$ that satisfy (17), belong to $\mathrm{co}(S)$.

From the defining conditions (12) and (14) of the set Σ, and since $v(X) = \begin{bmatrix}0 \ 0 \ 1\end{bmatrix}^T$, we have that

$$X_{11} + X_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Moreover, since matrices X_{11} and X_{22} are positive semi-definite, we have that the third and the sixth rows, as well as the third and the sixth columns of the matrix X are zero. Also, since all matrices from S are also positive semi-definite, the only matrices from S which can be summand in the convex combination making X, must have the same property, i.e. their third and sixth rows and columns are all zero. Therefore, from now on we can restrict only to the submatrix of X formed by the first, second, fourth and fifth rows and columns.

Then we have that the obtained matrix (still denoted by X) is of the following form:

$$X = \left[\begin{array}{cc|cc} a_1 & a_2 & b_1 & b_2+1 \\ a_2 & 1-a_1 & b_2 & -b_1 \\ \hline b_1 & b_2 & 1-a_1 & -a_2 \\ b_2+1 & -b_1 & -a_2 & a_1 \end{array}\right], \tag{18}$$

for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$. From the positive semi-definiteness (non-negativity of the principal minors), we obtain the following inequalities for the minors of the dimension two:

$$a_1(1-a_1) \geq a_2^2, \tag{19}$$

$$a_1^2 \geq (b_2+1)^2, \tag{20}$$

$$(1-a_1)^2 \geq b_2^2, \tag{21}$$

$$a_1(1-a_1) \geq b_1^2. \tag{22}$$

Also, by considering minors of the dimension one in the matrix (18), we have that

$$0 \leq a_1 \leq 1 \tag{23}$$

From (20), (21) and (23), we have $b_2 = a_1 - 1$. Moreover, the principal 3 by 3 minor of (18), gives:

$$(a_1 - 1)(a_2 + b_1)^2 \geq 0. \tag{24}$$

If $a_1 = 1$, then X automatically belongs to S, i.e. then $X \in S \in \mathrm{co}(S)$, which finishes our proof.

If $a_1 < 1$, then by (24), we must have $b_1 = -a_2$, and so our matrix X has the form

$$X = \left[\begin{array}{cc|cc} a_1 & a_2 & -a_2 & a_1 \\ a_2 & 1-a_1 & a_1-1 & a_2 \\ \hline -a_2 & a_1-1 & 1-a_1 & -a_2 \\ a_1 & a_2 & -a_2 & a_1 \end{array}\right], \tag{25}$$

with $a_1 \geq a_1^2 + a_2^2$.

On the other hand, matrices from S whose third and sixth rows and columns are zero, have the following form:

$$\left[\begin{array}{cc|cc} \cos^2\varphi & \sin\varphi\cos\varphi & -\sin\varphi\cos\varphi & \cos^2\varphi \\ \sin\varphi\cos\varphi & \sin^2\varphi & -\sin^2\varphi & \sin\varphi\cos\varphi \\ \hline -\sin\varphi\cos\varphi & -\sin^2\varphi & \sin^2\varphi & -\sin\varphi\cos\varphi \\ \cos^2\varphi & \sin\varphi\cos\varphi & -\sin\varphi\cos\varphi & \cos^2\varphi \end{array}\right],$$

with $\varphi \in \mathbb{R}$.

Hence we are left with proving that the point $(a_1, a_2) \in \mathbb{R}^2$ satisfying $a_1 \geq a_1^2 + a_2^2$, is in the convex hull of the set

$$K = \{(\cos^2\varphi, \sin\varphi\cos\varphi) \mid \varphi \in [0, 2\pi]\}.$$

Since K is a circle given by the equation $(x - \frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$, i.e. $x^2 + y^2 = x$, by (19) we have that the point (a_1, a_2) is inside the circle K. This finishes our proof. □

Remark 4. Above we have obtained that matrices of the form (25) with $a_1^2 + a_2^2 \leq a_1$ belong to $\mathrm{co}(S)$ (remember that the third and the sixth rows and columns are zero and we omit those rows and columns). By multiplying this matrix from the left by $\mathrm{diag}(I_2, -I_2)$ and from the right by $\mathrm{diag}(I_2, -I_2)$, we obtain that the matrices of the form

$$\left[\begin{array}{cc|cc} a_1 & a_2 & a_2 & -a_1 \\ a_2 & 1-a_1 & 1-a_1 & -a_2 \\ \hline a_2 & 1-a_1 & 1-a_1 & -a_2 \\ -a_1 & -a_2 & -a_2 & a_1 \end{array}\right], \tag{26}$$

with $a_1^2+a_2^2 \leq a_1$ also belong to $\mathrm{co}(S)$. This will be important in the following subsection.

3.2. *Case* $\mathrm{rank}(X_{11} + X_{22}) = 2$

Now, let $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ be a matrix from Σ such that $\mathrm{rank}\,(X_{11} + X_{22}) = 2$. In this case, from the conditions (12) and (14), we have that there exists an orthogonal matrix $P \in \mathbb{R}^{3\times 3}$ such that

$$P(X_{11}+X_{22})P^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{27}$$

and such that $PX_{11}P^T$ is diagonal. Therefore, without loss of generality we can assume that the matrix $X \in \Sigma$ satisfies that

$$X_{11}+X_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{28}$$

with X_{11} being a diagonal matrix. Also, by (14), the vector $v(X)$ has the form $v(X) = \begin{bmatrix} 0 \; 0 \; x \end{bmatrix}^T$, for some $|x| \le 1$.

Hence, we are left with proving that the positive semi-definite matrix of the form (we are not writing the third and the sixth rows and columns since they are all zero)

$$X = \left[\begin{array}{cc|cc} a_1 & 0 & b_1 & b_2+x \\ 0 & 1-a_1 & b_2 & -b_1 \\ \hline b_1 & b_2 & 1-a_1 & 0 \\ b_2+x & -b_1 & 0 & a_1 \end{array}\right], \tag{29}$$

belongs to $\mathrm{co}(S)$.

The case $|x| = 1$ we have already solved in Section 3.1, so we can assume that $-1 < x < 1$.

In fact, we shall show that the matrix X in (29) can be written as a convex combination of matrices of the forms (25) and (26), which, as we have already shown above, belong to $\mathrm{co}(S)$. Namely, we shall show that there exist $m_1, m_2, n_1, n_2 \in \mathbb{R}$, with $m_1^2 + m_2^2 \le m_1$, and $n_1^2 + n_2^2 \le n_1$, such that

$$X = \frac{1+x}{2}\left[\begin{array}{cc|cc} m_1 & m_2 & -m_2 & m_1 \\ m_2 & 1-m_1 & m_1-1 & m_2 \\ \hline -m_2 & m_1-1 & 1-m_1 & -m_2 \\ m_1 & m_2 & -m_2 & m_1 \end{array}\right] + \frac{1-x}{2}\left[\begin{array}{cc|cc} n_1 & n_2 & n_2 & -n_1 \\ n_2 & 1-n_1 & 1-n_1 & -n_2 \\ \hline n_2 & 1-n_1 & 1-n_1 & -n_2 \\ -n_1 & -n_2 & -n_2 & n_1 \end{array}\right]. \tag{30}$$

Straightforward computation gives the unique solution of (30):

$$m_1 = \frac{a_1+b_2+x}{1+x},$$

$$m_2 = -\frac{b_1}{1+x},$$

$$n_1 = \frac{a_1-b_2-x}{1-x},$$

$$n_2 = \frac{b_1}{1-x}.$$

Since $|x| < 1$, the right-hand side of the expression (30) is a convex combination of two matrices of the forms (25) and (26), respectively. Hence, in order to use the result from Section 3.1 that these two matrices are from $\mathrm{co}(S)$, we are left with proving that $m_1^2 + m_2^2 \le m_1$ and $n_1^2 + n_2^2 \le n_1$, i.e.

$$b_1^2 \le \min\{(a_1 + b_2 + x)(1 - a_1 - b_2), (a_1 - b_2 - x)(1 - a_1 + b_2)\}. \tag{31}$$

In order to prove (31), we use the fact that $X \succcurlyeq 0$, and in particular that its determinant is nonnegative, which gives:

$$\begin{aligned} &b_1^4 - 2(a_1(1 - a_1) - b_2(b_2 + x))b_1^2 \\ &\quad + (1 - a_1 - b_2)(1 - a_1 + b_2)(a_1 - b_2 - x)(a_1 + b_2 + x) \ge 0. \end{aligned} \tag{32}$$

Denote by

$$A = a_1(1 - a_1) - b_2(b_2 + x)$$

and

$$B = (1 - a_1 - b_2)(1 - a_1 + b_2)(a_1 - b_2 - x)(a_1 + b_2 + x).$$

Then we have

$$A^2 - B = (a_1 b_2 - (1 - a_1)(b_2 + x))^2,$$

and so (32) is equivalent to

$$b_1^2 \le A - \sqrt{A^2 - B} \quad \text{or} \quad b_1^2 \ge A + \sqrt{A^2 - B}. \tag{33}$$

However, from the non-negativity of the principal 3 by 3 minors of the matrix X from (29), we obtain that

$$b_1^2 \le a_1(1 - a_1) - \max\left\{\frac{a_1 b_2^2}{1 - a_1}, \frac{(1 - a_1)(b_2 + x)^2}{a_1}\right\}.$$

The maximum of the two nonnegative numbers is always greater or equal to their geometric mean, and so we obtain

$$b_1^2 \le a_1(1 - a_1) - |b_2(b_2 + x)| \le A.$$

Therefore, only the first inequality in (33) is valid, and it reads

$$b_1^2 \le a_1(1 - a_1) - b_2(b_2 + x) - |a_1 b_2 - (1 - a_1)(b_2 + x)|.$$

The last is equivalent to (31), which finishes our proof. □

4. Applications and experiments

Our result is of the highest interest for structure from motion (SfM) problems in computer vision. In fact, in [3] it already proved to be extremely effective in solving SfM problems for nonrigid (deformable) shapes.

Here we show that it can also be used for Procrustes-like problems on $\mathsf{O}(3,2)$.

Consider the following Procrustes problem:

Let $B \in \mathbb{R}^{3\times 2}$ and $A \in \mathbb{R}^{3\times 3}$

$$\underset{\text{subject to } Q\in\mathsf{O}(3,2)}{\text{minimize}} \|B - AQ\|^2 \tag{34}$$

Problem (34) can be regarded as a typical subproblem in SfM algorithms (e.g. see [3]).

Problem (34) is equivalent to

$$\underset{\text{subject to } X\in\mathcal{X}}{\text{minimize}} \operatorname{Tr}\left(\begin{bmatrix} \mathcal{A} & \frac{1}{2}c \\ \frac{1}{2}c^T & 0 \end{bmatrix} X\right) \tag{35}$$

where $\mathcal{A} := I_2 \otimes A^T A$, $c := 2\operatorname{vec}(A^T B)$ and

$$\mathcal{X} = \left\{ X = \begin{bmatrix} X_{11} & X_{12} & x_1 \\ X_{21} & X_{22} & x_2 \\ x_1^T & x_2^T & 1 \end{bmatrix} \succcurlyeq 0 \,:\, \operatorname{Tr}(X_{ii}) = 1, \operatorname{Tr}(X_{12}) = 0, \operatorname{rank} X = 1 \right\}.$$

Now, the "standard" convex relaxation for (35) consists in dropping the rank constraint from $\mathcal{X}$, thus turning (35) into a (relaxed) SDP.

Let $X^\star$ be the solution obtained by solving a particular instance of the relaxed SDP. Note that $X^\star$ is also a solution of (35) if it happens that $\operatorname{rank} X^\star = 1$ or, equivalently,

$$\lambda_{\max}(X^\star) = 3, \tag{36}$$

where $\lambda_{\max}(X)$ denotes the maximum eigenvalue of a symmetric matrix X.

Here, we propose a tighter relaxation, by capitalizing on our result. First, note that (34) can be rewritten as

$$\underset{\text{subject to } q=\operatorname{vec}(Q)}{\text{minimize}} \operatorname{Tr}\left(\mathcal{A}qq^T\right) - |q^T c| \tag{37}$$

with the understanding that Q runs through the Stiefel matrices $\mathsf{O}(3,2)$. Problem (37) is, in turn, equivalent to

$$\underset{\text{subject to } X\in S}{\text{minimize}} \operatorname{Tr}(\mathcal{A}X) - \sqrt{c^T X c} \tag{38}$$

Recall that $S \subset \mathbb{R}^{6\times 6}$ was defined in (4)–(6). We now propose to relax the nonconvex constraint set S in (38) to the convex set Σ defined in Conjecture 3. This turns (38) into a (relaxed) convex problem which, in fact, is easily reformulated into a SDP.

Let $X^\star$ be a solution of this relaxed SDP. Then, $X^\star$ is also a solution of (38) if $\operatorname{rank} X^\star = 1$ or, equivalently,

$$\lambda_{\max}(X^\star) = 2. \tag{39}$$

5. Numerical experiments

We report some numerical experiments to illustrate the tightness of the convex approximation obtained by the two aforementioned methods ("standard" and ours). We generated more than 1000 random instances of problem (34) (in each instance, the entries of A and B were independently drawn from a zero-mean unit-variance Gaussian distribution). For instance k ($k = 1, 2, \dots, 1000$), we solved the "standard" convex relaxation of (35) (i.e., we dropped the rank constraint from $\mathcal{X}$) and our convex relaxation of (38) (i.e., we replaced S with Σ). Let $X_{1k}^\star$ and $X_{2k}^\star$ denote the respective obtained solutions on instance k. We then checked if $X_{1k}^\star$ and $X_{2k}^\star$ also solved the associated nonconvex problems (35) and (38) (both equivalent to (34)), i.e., we verified if (36) and (39) hold (we used the tests, $\lambda_{\max}(X_{1k}^\star) \geq 2.9999$ and $\lambda_{\max}(X_{2k}^\star) \geq 1.9999$, respectively). For the "standard" relaxation, only 48% of the instances turned out to be exact, whereas our method was exact in 99.9% of the instances.

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