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Global asymptotic stabilization of spherical orientation by synergistic hybrid feedback with application to reduced attitude synchronization

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***Abstract*— In this paper, we develop a hybrid controller for global asymptotic stabilization on the n-dimensional sphere ($\mathbb{S}^n$) using synergistic potential functions. These consist of a collection of potential functions on $\mathbb{S}^n$ that induce a gradient descent controller during flows of the hybrid closed-loop system and a switching law that, at undesired equilibrium points of the gradient vector field, jumps to the lowest value among all the potential functions in the collection. We show that the proposed controller can be used for global reduced attitude synchronization, i.e., given a network of rigid-bodies, the proposed synergistic hybrid feedback can be used to globally synchronize a reference direction of each agent within a global but unknown inertial reference frame. We study this application for a network of three vehicles by means of simulation results.**

I. INTRODUCTION

In this paper, we develop new hybrid feedback tools to tackle the limitations to global asymptotic stabilization of spherical orientation and reduced attitude synchronization by continuous and discontinuous feedback. Namely, it was shown in [1] that there is no continuous feedback law that is able to globally asymptotically stabilize a system evolving on a compact manifold due to topological obstructions and it was shown in [2] that, if it is not possible to globally asymptotically stabilize a set by means of continuous feedback, then it is not possible to robustly globally asymptotically stabilize it with discontinuous feedback either. In particular, we resort to synergistic hybrid feedback strategies in order to address the aforementioned problems. This class of hybrid controllers relies on the existence of a collection of continuously differentiable functions that are positive definite relative to a setpoint and, at points other than the desired setpoint where its gradient vanishes or is not defined, it is possible to find another function in the collection with a lower value. In this way, global asymptotic stabilization of the given setpoint is enabled by a combination of switching near undesired equilibrium points and gradient-based feedback.

Over the past few years, we have witnessed the steady development of synergistic hybrid feedback and its application to multiple problems in control, such as: global asymptotic stabilization of systems evolving on compact manifolds (see e.g., [3], [4]); global exponential stabilization on $\mathrm{SO}(3)$ and $\mathbb{S}^n$ (c.f. [5], [6] and [7]); trajectory tracking for thrust-vectored vehicles [8]; output-based control of rigid-body vehicles (see e.g. [9]); and global synchronization of rigid-body attitude by unit-quaternion feedback [10].

Synchronization of multi-agent systems corresponds to the situation where all the agents agree on the value of some variable. The design of synchronization strategies for a network of vehicles allows them to find a common reference frame when no inertial frame is given, enabling many applications such as: cooperative surveillance of an area [11], cooperative patrolling [12] and bearing-based formation control [13]. In particular, some of these strategies require the design of synchronization strategies for networks of rigid-body vehicles, explaining the extensive literature in this field, which includes [14], [10] and [15], for example. The problem of reduced attitude synchronization amounts to the synchronization of a single reference direction for each rigid-body agent and it has also received some attention in the last few years, as evidenced by the works [16], [17] and [18]. However, the aforementioned solutions are still limited to almost global synchronization.

In this paper, we further extend the existing body of work on synergistic hybrid feedback by constructing a new kind of synergistic potential function on $\mathbb{S}^n$ that does not necessarily consist on a finite collection of functions such as the ones in [19]. This relaxation was already present in [7] for the particular case of centrally synergistic potential functions on $\mathbb{S}^n$, but it is extended in this paper for the non-central case as well. We show that it is possible to derive a hybrid controller that globally asymptotically stabilizes a given setpoint $r \in \mathbb{S}^n$ from a synergistic potential function relative to r. Moreover, we show that the proposed synergistic potential function can be applied to the problem of global reduced attitude synchronization on a network of rigid-bodies. Specifically, given a network of rigid-body agents that is modeled by an undirected graph, we provide a set of sufficient conditions for the global reduced attitude synchronization and we show that it is possible to meet those conditions if the graph is a tree.

This paper is organized as follows: in Section II we present the notation and some definitions that are used throughout the paper; in Section III we introduce the concept of synergistic potential function on $\mathbb{S}^n$ and we show that it induces a hybrid controller that renders a given setpoint globally asymptotically stable for the closed-loop system; in Section IV we provide a new construction for this class of synergistic potential functions; in Section V we show how

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the proposed synergistic potential function can be applied to the problem of global reduced attitude synchronization; in Section VI we provide some simulation results that illustrate our findings and; in Section VII we present the conclusions of this work.

II. Preliminaries

A. Notation

The symbol $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, equipped with the inner product $\langle u, v\rangle := u^\top v$, defined for each $(u, v) := [u^\top\ v^\top]^\top \in \mathbb{R}^n \times \mathbb{R}^n$. Moreover, the norm of a vector $x \in \mathbb{R}^n$ is given by $|x| := \sqrt{\langle u, v\rangle}$. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. The $n \times n$ identity matrix is represented by I_n. The symbol $0_{n\times m}$ denotes a $n \times m$ matrix of zeros. The image of a matrix $A \in \mathbb{R}^{n\times m}$ is given by $\mathrm{Image}(A) := \{y \in \mathbb{R}^n : \exists x \in \mathbb{R}^m\ y = Ax\}$. Given $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$, $A \otimes B$ denotes the Kronecker product between A and B. The interior of a set S is denoted by $\mathrm{int}(S)$.

Given a compact set Q, a function $V : \mathbb{S}^n \times Q \to \mathbb{R}$ is continuously differentiable, also written as $V \in C^1(\mathbb{S}^n \times Q, \mathbb{R})$ if, for each $q \in Q$, the gradient

$$\nabla V(x, q) := \left[\frac{\partial V}{\partial x_1}(x, q) \quad \dots \quad \frac{\partial V}{\partial x_{n+1}}(x, q)\right]^\top$$

is defined for all $x \in \mathbb{S}^n$ and is continuous.

A network can be modeled by a graph $\mathsf{G} := (\mathsf{V}, \mathsf{E})$, where V is a non-empty finite set, E is a relation on V and $|\mathsf{V}|$, $|\mathsf{E}|$ denote the cardinality of V and E, respectively.

B. Hybrid Systems

A hybrid system $\mathcal{H}$ with state space $\mathbb{R}^n$ is defined as follows:

$$\begin{aligned} \dot{\xi} &\in F(\xi) \quad \xi \in C \\ \xi^+ &\in G(\xi) \quad \xi \in D \end{aligned}$$

where $\xi \in \mathbb{R}^n$ is the state, $C \subset \mathbb{R}^n$ is the flow set, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the flow map, $D \subset \mathbb{R}^n$ denotes the jump set, and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the jump map. A solution ξ to $\mathcal{H}$ is parametrized by (t, j), where t denotes ordinary time and j denotes the jump time, and its domain $\mathrm{dom}\,\xi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain: for each $(T, J) \in \mathrm{dom}\,\xi$, $\mathrm{dom}\,\xi \cap ([0, T] \times \{0, 1, \dots J\})$ can be written in the form $\cup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_J$, where $I_j := [t_j, t_{j+1}]$ and the t_j's define the jump times. A solution ξ to a hybrid system is said to be *maximal* if it cannot be extended by flowing nor jumping and *complete* if its domain is unbounded. The projection of solutions onto the t direction is given by $\xi\downarrow_t(t) := \xi(t, J(t))$ where $J(t) := \max\{j : (t, j) \in \mathrm{dom}\,\xi\}$. The distance of a point $\xi \in \mathbb{R}^n$ to a closed set $\mathcal{A} \subset \mathbb{R}^n$ is given by $|\xi|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |y - \xi|$. The definitions of global uniform pre-asymptotic stability, pre-asymptotic stability, strong pre-forward invariance, outer and upper semicontinuity of a set-valued map, local boundedness are all used in this paper and can be found in [20].

III. Global Asymptotic Stabilization on $\mathbb{S}^n$ by Synergistic Hybrid Feedback

In this section, we develop a hybrid controller for global asymptotic stabilization of a system on the n-dimensional sphere $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : x^\top x = 1\}$, whose dynamics are described by

$$\dot{x} = \Pi(x)\omega \tag{1}$$

where $x \in \mathbb{S}^n$ denotes the state of the system and $\omega \in \mathbb{R}^{n+1}$ is the input. The operator $\Pi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1\times n+1}$ is given by $\Pi(x) := I_{n+1} - xx^\top$ and it is such that $\mathrm{Image}(\Pi(x)) \subset \mathbb{R}^{n+1}$ is the tangent space to $\mathbb{S}^n$ at x. The hybrid controller that we propose is built from synergistic potential functions on $\mathbb{S}^n$, which we define next.

Definition 1. *Given $r \in \mathbb{S}^n$ and a compact set Q, a function $V \in C^1(\mathbb{S}^n \times Q, \mathbb{R})$ is a synergistic potential function candidate relative to r if there exists a unique closed subset P of Q such that V is positive definite relative to*

$$\mathcal{A} := \{r\} \times P, \tag{2}$$

i.e., $V(x, q) \geq 0$ for each $(x, q) \in \mathbb{S}^n \times Q$ and $V(x, q) = 0$ if and only if $(x, q) \in \mathcal{A}$.

The set of critical points of a function V that is a synergistic potential function candidate relative to $r \in \mathbb{S}^n$ is given by

$$\mathcal{E} := \{(x, q) \in \mathbb{S}^n \times Q : \Pi(x)\nabla V(x, q) = 0\} \tag{3}$$

and it corresponds to the pairs $(x, q) \in \mathbb{S}^n \times Q$ where the gradient of V is orthogonal to the tangent space to $\mathbb{S}^n$. This includes, in particular, the set $\mathcal{A}$ given in (2), as proved next.

Lemma 1. *Given $r \in \mathbb{S}^n$ and a synergistic potential function candidate relative to r, denoted by $V \in C^1(\mathbb{S}^n \times Q, \mathbb{R})$, the following hold:*

1) *The set $\mathcal{A}$ in* (2) *is a compact subset of $\mathcal{E}$ in* (3)*;*
2) *The function*

$$\varrho(x) := \min\{V(x, s) : s \in Q\} \quad \forall x \in \mathbb{S}^n, \tag{4}$$

is continuous;

3) *The set-valued map*

$$\nu(x) := \arg\min\{V(x, s) : s \in Q\} \quad \forall x \in \mathbb{S}^n \tag{5}$$

is outer semicontinuous.

Proof. This result follows from [21, Theorem 9.14] and [20, Lemma 5.15]. □

The set $\mathcal{E}\backslash\mathcal{A}$ corresponds to undesired equilibrium points of the gradient-based feedback

$$\omega(x, q) := -\nabla V(x, q) \quad \forall (x, q) \in \mathbb{S}^n \times Q. \tag{6}$$

The rationale for synergistic hybrid feedback consists of switching to the gradient-based feedback of a function $x \mapsto V(x, q')$ with $(x, q') \notin \mathcal{E}$ while satisfying $V(x, q) - V(x, q') > 0$ so that V decreases its value during jumps. To trigger controller switching, we monitor the so-called *synergy gap*, which is defined as follows:

$$\mu_V(x, q) := V(x, q) - \varrho(x) \quad \forall (x, q) \in \mathbb{S}^n \times Q \tag{7}$$

with ϱ given in (4).

Definition 2. *A synergistic potential function candidate relative to* r*, denoted by* $V \in C^1(\mathbb{S}^n \times Q, \mathbb{R})$*, is a synergistic potential function relative to* r *if there exists* $\delta > 0$ *such that* $\mu_V(x,q) > \delta$ *for each* $(x,q) \in (\mathcal{E} \cup \mathcal{B}) \backslash \mathcal{A}$*, where*

$$\mathcal{B} := \{r\} \times Q. \tag{8}$$

The definition of a synergistic potential function on $\mathbb{S}^n$ given in Definition 2 is akin to that of [19], with the key difference that Q is considered to be compact rather than finite. On the other hand, the centrally synergistic potential functions defined in [7] are particular cases of Definition 2 that correspond to the case where $\mathcal{B} = \mathcal{A}$. A synergistic potential function V relative to r induces the following hybrid controller for global asymptotic stabilization of $\mathcal{A}$ for (1):

$$\begin{aligned} \dot{q} &= 0 && (x,q) \in C := \{(x,q) \in \mathbb{S}^n \times Q : \mu_V(x,q) \leq \delta\} \\ q^+ &\in \nu(x) && (x,q) \in D := \{(x,q) \in \mathbb{S}^n \times Q : \mu_V(x,q) \geq \delta\} \end{aligned} \tag{9}$$

with output $(x,q) \mapsto \omega(x,q)$ given in (6). The closed-loop system $\mathcal{H} := (C, F, D, G)$ resulting from the interconnection between (1) and (9) is given by

$$\begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = F(x,q) := \begin{bmatrix} -\Pi(x)\nabla V(x,q) \\ 0 \end{bmatrix} \quad (x,q) \in C \tag{10a}$$

$$\begin{bmatrix} x^+ \\ q^+ \end{bmatrix} \in G(x,q) := \begin{bmatrix} x \\ \nu(x) \end{bmatrix} \quad (x,q) \in D \tag{10b}$$

and it satisfies the so-called *hybrid basic conditions*, as proved below.

Lemma 2. *Given a synergistic potential function candidate* V *relative to* r*, the closed-loop hybrid system* $\mathcal{H}$ *in* (10) *satisfies the following: 1)* C *and* D *are closed; 2)* F *is outer semicontinuous and locally bounded relative to* C*, and* $F(x,q)$ *is convex for each* $(x,q) \in C$*; 3)* G *is outer semicontinuous relative to* D *and locally bounded relative to* D*.*

Proof. C and D are closed because they are the pre-images of the closed sets (c.f. [22]). The function F is continuous, hence it is outer semicontinuous, locally bounded relative to $\mathbb{S}^n \times Q$ (c.f. [23]). Moreover, it is convex for each $(x,q) \in \mathbb{S}^n \times Q$ because it is single-valued. It follows from Lemma 1 that G is outer semicontinuous relative to $\mathbb{S}^n \times Q$. It is locally bounded relative to $\mathbb{S}^n \times Q$ because $G_1(x,q) = x$ is continuous and ν takes values on a compact set. □

These conditions are important in establishing the stability results presented in this section. Moreover, they guarantee nominal robustness to small measurement noise. The reader is referred to [20, Chapter 7] for more information.

In the following lemma, we present some preliminary results that are important to the stability results presented in this section.

Lemma 3. *Given a synergistic potential function* V *relative to* r*, the following hold:*

$$G(D) \cap D = \emptyset \tag{11a}$$

$$((\mathcal{E} \cup \mathcal{B}) \backslash \mathcal{A}) \cap C = \emptyset. \tag{11b}$$

The condition (11a) together with Lemma 2 ensures that, for each solution to (10), there is a positive lower bound on the time between jumps, as demonstrated in [24]. The condition (11b) not only ensures that all undesired critical points of the gradient-based feedback in $\mathcal{E} \backslash \mathcal{A}$ belong to the jump set, but also that every point in $\mathcal{B} \backslash \mathcal{A}$ belongs to the jump set as well. Finally, we show that each maximal condition to (10) is complete, which is pivotal to global asymptotic stabilization of $\mathcal{A}$ for $\mathcal{H}$.

Lemma 4. *Given a synergistic potential function* V *relative to* r*, each maximal solution to* $\mathcal{H}$ *given in* (10) *is complete.*

Lemma 5. *Given a synergistic potential function* V *relative to* r*, the set* $\mathcal{A}$ *given in* (2) *is uniformly globally asymptotically stable for* $\mathcal{H}$ *given in* (10)*.*

Proof. Since $\mathcal{A}$ is compact, we prove that $\mathcal{A}$ is globally asymptotically stable for $\mathcal{H}$ using [20, Corollary 8.9].

Since $\mathcal{H}$ satisfies [20, Assumption 6.5] as shown in Lemma 2, it follows from [20, Theorem 6.8] that $\mathcal{H}$ is nominally well-posed. It follows from [20, Theorem 7.12] and the completeness of solutions that $\mathcal{A}$ is globally $\mathcal{KL}$ asymptotically stable which, by virtue of [20, Theorem 3.40], is equivalent to uniform global asymptotic stability of $\mathcal{A}$ for $\mathcal{H}$. □

Given a solution $(t,j) \mapsto (x,q)(t,j)$, Lemma 5 does not ensure that, if $x(t^\star, j^\star) = r$ for some $(t^\star, j^\star) \in \operatorname{dom}(x,q)$, then $x(t,j) = r$ for all $(t,j) \in \operatorname{dom}(x,q)$ satisfying $t + j \geq t^\star + j^\star$. To ensure that this is the case, we provide the following result.

Theorem 1. *Given a synergistic potential function* V *relative to* r*, the set* $\mathcal{B}$ *given in* (8) *is globally asymptotically stable for* $\mathcal{H}$ *given in* (10)*.*

Proof. Since $\mathcal{B}$ is compact, strongly forward invariant and $\mathcal{H}$ is nominally well-posed (see the proof of Lemma 5), it follows from [20, Proposition 7.5] that $\mathcal{B}$ is globally asymptotically stable for $\mathcal{H}$. □

IV. Constructing a Synergistic Potential Function on $\mathbb{S}^n$

Let $Q := Q_r \cup \{r\}$ with

$$Q_r := \{q \in \mathbb{S}^n : q^\top r = \gamma\} \tag{12}$$

and $\gamma \in (-1, 1)$ and let

$$V(x,q) := \alpha_q + \beta_q(1 - q^\top x) \quad \forall (x,q) \in \mathbb{S}^n \times Q \tag{13}$$

where

$$\alpha_q := \begin{cases} \alpha & \text{if } q \in Q_r \\ 0 & \text{otherwise} \end{cases}, \quad \beta_q := \begin{cases} \beta & \text{if } q \in Q_r \\ 1 & \text{otherwise} \end{cases} \tag{14}$$

with $\alpha, \beta > 0$. The function V in (13) is continuously differentiable and its gradient in given by $\nabla V(x,q) = -\beta_q q$ for each $(x,q) \in \mathbb{S}^n \times Q$. Since V is positive definite relative to $\mathcal{A}$ in (2) with $P = \{r\}$, it follows that it is a synergistic potential function candidate relative to r. Using this construction, we have that $\mathcal{A}$ in (2) and $\mathcal{B}$ in (8) are given by $\mathcal{A} = \{r\} \times \{r\}$ and $\mathcal{B} = \{r\} \times Q$, respectively. The critical points of V belong to the set $\mathcal{E}$ in (3) which is given by $\mathcal{E} = \{(x,q) \in \mathbb{S}^n \times Q : x = \pm q\}$. It follows that $(\mathcal{E} \cup \mathcal{B}) \backslash \mathcal{A}$ is given by $(\mathcal{E} \cup \mathcal{B}) \backslash \mathcal{A} = \{(x,q) \in \mathbb{S}^n \times Q_r : x = \pm q\} \cup \{r\} \times Q_r \cup \{-r\} \times \{r\}$.

The minimum and the minimizer of V with respect to q are given in the following proposition.

Proposition 1. *Given $Q = Q_r \cup \{r\}$ and V in* (13)*, the functions ϱ in* (4) *and ν in* (5) *are given by*

$$\varrho(x) = \min\{1 - r^\top x, \\ \alpha + \beta(1 - \gamma x^\top r - \sqrt{1-\gamma^2}\,|\Pi(r)x|)\} \tag{15a}$$

$$\nu(x) = \arg\min\left\{\alpha_q + \beta_q(1 - q^\top x) : \\ q \in \left\{\gamma r + \sqrt{1-\gamma^2}\frac{\Pi(r)x}{|\Pi(r)x|}, r\right\}\right\} \tag{15b}$$

for each $x \in \mathbb{S}^n \backslash \{-r, r\}$, respectively, and

$$\varrho(r) = 0, \quad \varrho(-r) = \min\{2, \alpha + \beta(1-\gamma)\}, \tag{16a}$$

$$\nu(r) = r, \quad \nu(-r) = \arg\min\{\alpha_q + \beta_q(1 + r^\top q) : q \in Q\}. \tag{16b}$$

Proof. We derive this result using standard algorithms for constrained optimization using Lagrange multipliers. □

In order to qualify as a synergistic potential function in $\mathbb{S}^n$, the function V must meet the criteria imposed by Definition 2, which restricts the range of the parameters $\alpha, \beta > 0$, as shown in the following proposition.

Proposition 2. *Given $r \in \mathbb{S}^n$ and $Q = Q_r \cup \{r\}$, the function V given in* (13) *is a synergistic potential function relative to r if and only if*

$$\beta < 1 \quad \text{and} \quad 1 - \gamma < \alpha < 2 - \beta(1+\gamma). \tag{17}$$

Moreover, if (17) *is satisfied, then V a synergistic potential function relative to r with synergy gap exceeding $\delta \in (0, \min\{\delta_i : i \in \{1,2,3,4\}\})$ with*

$$\delta_1 := \alpha - 1 + \gamma$$
$$\delta_2 := \max\{2\beta(1-\gamma^2), \alpha + 2\beta - 1 - \gamma\}$$
$$\delta_3 := \alpha + \beta(1-\gamma)$$
$$\delta_4 := 2 - \alpha - \beta(1+\gamma).$$

Proof. The conditions (17) and the parameters 18 are obtained from the evaluation of (7) at $(\mathcal{E} \cup \mathcal{B}) \backslash \mathcal{A}$. □

V. Global Reduced attitude synchronization

Let us consider a network of rigid-body agents that is modeled by a undirected graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ without self-loops where each agent is characterized by the kinematics

$$\dot{R}_i = R_i S(\omega_i), \quad \forall i \in \{1, \ldots, |\mathsf{V}|\}$$

with $R_i \in \mathrm{SO}(3) := \{R \in \mathbb{R}^{3\times3} : R^\top R = I_3, \det(R) = 1\}$ representing its attitude, $\omega_i \in \mathbb{R}^3$ its input angular velocity in body-fixed coordinates, and $S : \mathbb{R}^3 \to \mathbb{R}^{3\times3}$ is such that $S(u)v = u \times v$ for each $u, v \in \mathbb{R}^3$.

In this section, we design a controller for global asymptotic synchronization of the reduced attitude of the rigid-bodies in the network G, i.e., given reference directions $r_i \in \mathbb{S}^2$ for $i \in \{1, \ldots, |\mathsf{V}|\}$ where $|\mathsf{V}|$ corresponds the number of nodes in G and letting $(R, x_c) := (R_1, \ldots, R_{|\mathsf{V}|}, x_c) \in \mathcal{S} := \mathrm{SO}(3)^{|\mathsf{V}|} \times \mathcal{X}$ where $x_c \in \mathcal{X}$ represents the state of the hybrid controller, the problem of reduced attitude synchronization amounts to the global asymptotic stabilization of

$$\mathcal{B}_{\mathsf{G}} := \{(R, x_c) \in \mathcal{S} : \\ R_i r_i = R_j r_j \text{ for all } i, j \in \{1, \ldots, |\mathsf{V}|\}\} \tag{19}$$

for the closed-loop system using only relative measurements between neighboring agents.

It was shown in [17, Theorem 5.1] that it is possible to achieve almost global reduced attitude synchronization for networks that can be modelled by tree graphs using distance functions, i.e., functions that can be written in the form $(R_i, R_j) \mapsto f(1 - r_i^\top R_i^\top R_j r_j)$ for some smooth function $f : [0,2] \to \mathbb{R}$ and $i \neq j$, such as the function $(R_i, R_j) \mapsto V(R_i^\top R_j r_j, r_i)$ with V given in (13). Unfortunately, it is not possible to construct a synergistic potential function from a collection of distance functions, as proved next.

Lemma 6. *Given $r \in \mathbb{S}^n$ and a compact set Q, there does not exist a collection $\{V_q\}_{q\in Q}$ where $x \mapsto V_q(x) = f_q(1 - r^\top x)$ for some continuously differentiable function $f_q : [0,2] \to \mathbb{R}$ such that*

$$V(x, q) = V_q(x) \quad \forall (x, q) \in \mathbb{S}^n \times Q$$

is a synergistic potential function on relative to r.

To remedy this issue, we resort to the synergistic potential function introduced in Section IV. More specifically, consider a subgraph $\mathsf{G}' := (\mathsf{V}', \mathsf{E}')$ of G with the property

$$\mathsf{E}' \subset L := \{(i,j) \in \mathsf{E} : i < j\} \tag{20}$$

such that, for each edge $(i,j) \in \mathsf{E}'$, the i-th agent stores an internal variable

$$q_{ij} \in Q_{ij} := Q_{r_i} \cup \{r_i\} \tag{21}$$

with Q_{r_i} given in (12) for some $\gamma \in (-1, 1)$ and satisfying $\dot{q}_{ij} = 0$. Under this construction, the controller state variable x_c corresponds to a ordered tuple that contains the logic variables q_{ij}, where the order is given by a one-to-one function $\bar{\kappa} : L \to M := \{1, \ldots, |\mathsf{E}|\}$ that labels each edge of G. In other words, $x_c := (q_{k_1}, \ldots, q_{k_{|\mathsf{E}'|}}) \in \mathcal{X} := Q_{k_1} \times \ldots \times Q_{k_{|\mathsf{E}'|}}$, where $q_{k_\ell} = q_{ij}$ and $Q_{k_\ell} = Q_{ij}$ for $(i,j) \in \bar{\kappa}^{-1}(k_\ell)$ and $k_a < k_b$ for any $a, b \in \{1, \ldots, |\mathsf{E}|\}$ satisfying $a < b$. The labeling function $\bar{\kappa}$ can be used to construct the incidence matrix $B \in \mathbb{R}^{|\mathsf{V}|\times|\mathsf{E}|}$ of the graph G as follows: for every $k \in M$ and for $(i,j) = \bar{\kappa}^{-1}(k)$, $B_{ik} = 1$, $B_{jk} = -1$ and $B_{\ell k} = 0$ for all $\ell \in \mathsf{V} \backslash \{i, j\}$.

We define the feedback law of the i-th agent in the network as follows:

$$\omega_i(R, x_c) := -\sum_{j \in \{j \in \mathsf{V} : \{i,j\} \in \mathsf{E}\}} S(R_i^\top R_j \psi_{ji})\psi_{ij} \tag{22}$$

for each $(R, x_c) \in \mathcal{S}$ and $i \in \mathsf{V}$, where

$$\psi_{ij} := \begin{cases} \beta_{q_{ij}} q_{ij} & \text{if } (i,j) \in \mathsf{E}' \\ r_j & \text{otherwise} \end{cases}$$

for each $\{i, j\} \in \mathsf{E}$, where $\beta_{q_{ij}}$ is given in (14). Note that, for each $(i,j) \in \mathsf{E}'$, the feedback law (22) corresponds to the gradient-based feedback of the synergistic potential function V_{ij}, otherwise it corresponds to the gradient-based feedback of $R_i^\top R_j r_j \mapsto h(R_i^\top R_j r_j, r_i) := 1 - r_i^\top R_i^\top R_j r_j$. In the next lemma, we show that the feedback law has a linear dependence on the error vector $e(R, x_c) := (e_1(R, x_c), \ldots, e_{|\mathsf{E}|}(R, x_c)) \in \mathbb{R}^{3|\mathsf{E}|}$, where

$$e_k(R, x_c) := \begin{cases} \beta_{q_{ij}} S(R_i q_{ij}) R_j r_j & \text{if } (i,j) \in \bar{\kappa}^{-1}(k) \cap \mathsf{E}' \\ S(R_i r_i) R_j r_j & \text{if } (i,j) \in \bar{\kappa}^{-1}(k) \backslash \mathsf{E}' \end{cases}$$

for each $(R, x_c) \in \mathcal{S}$ and $k \in \{1, \ldots, |\mathsf{E}|\}$.

Lemma 7. *Given a graph $\mathsf{G} := (\mathsf{V}, \mathsf{E})$ and a subgraph G' of G satisfying (20), the function*

$$\omega(R, x_c) := (\omega_1(R, x_c), \ldots, \omega_{|\mathsf{V}|}(R, x_c))$$

defined for each $(R, x_c) \in \mathcal{S}$ satisfies

$$\omega(R, x_c) = \operatorname{diag}(R)^\top (B \otimes I_3) e(R, x_c),$$

where $B \in \mathbb{R}^{|\mathsf{V}| \times |\mathsf{E}|}$ denotes the incidence matrix of G and $\operatorname{diag}(R) \in \mathbb{R}^{3|\mathsf{V}| \times 3|\mathsf{V}|}$ is a block diagonal matrix, whose diagonal blocks are given by $R_i \in \mathrm{SO}(3)$ for each $i \in \{1, \ldots, |\mathsf{V}|\}$.

Following the synergistic hybrid feedback approach, let us define

$$G_{ij}(R) := \{x_c \in \mathcal{X} : q_{ij} \in \nu_{ij}(R_i^\top R_j r_j)\}$$
$$\forall (R, x_c) \in D_{ij} := \{(R, x_c) \in \mathcal{S} : \mu_{V_{ij}}(R_i^\top R_j r_j, q_{ij}) \geq \delta\}$$

and

$$C_{ij} := \{(R, x_c) \in \mathcal{S} : \mu_{V_{ij}}(R_i^\top R_j r_j, q_{ij}) \leq \delta\}$$

for each $(i,j) \in \mathsf{E}'$, where the function $V_{ij} : \mathbb{S}^2 \times Q_{ij} \to \mathbb{R}_{\geq 0}$ is given by (13) with parameters α and β satisfying (17) so that V_{ij} is synergistic potential function relative to $r_i \in \mathbb{S}^2$ by virtue of Proposition 2, $\mu_{V_{ij}}$ is given by (7), ν_{ij} is given by (15b) and $\delta \in (0, \min\{\delta_i : i \in \{1,2,3,4\}\})$ with δ_i given in (18) for each $i \in \{1,2,3,4\}$. The hybrid closed-loop system is given by

$$\begin{cases} \dot{R} = \begin{bmatrix} R_1 S(\omega_1(R, x_c)) \\ \vdots \\ R_{|\mathsf{V}|} S(\omega_{|\mathsf{V}|}(R, x_c)) \end{bmatrix} & (R, x_c) \in \bigcap_{(i,j) \in \mathsf{E}'} C_{ij}, \\ \dot{x_c} = 0 \end{cases}$$
$$\begin{cases} R^+ = R & (R, x_c) \in \bigcup_{(i,j) \in \mathsf{E}'} D_{ij}. \\ x_c^+ \in \bigcup_{(i,j) \in \mathsf{E}'} G_{ij}(R) \end{cases} \tag{23}$$

In the next theorem, we show that, under appropriate assumptions on E', every undesired equilibrium point of the closed-loop system can be avoided by switching the gradient-based feedback associated with a synergistic potential function. Note that the switching of the logic variable q_{ij} is internal to the i-th agent for each $i \in \mathsf{V}'$. However, neighbors of $i \in \mathsf{V}'$ that are connected through an edge $(i,j) \in \mathsf{E}'$ must be aware of changes to q_{ij} in order for the synchronization strategy to work.

Theorem 2. *Given the network $\mathsf{G} := (\mathsf{V}, \mathsf{E})$ with incidence matrix B and the subgraph G' satisfying (20), if, for each $(R, x_c) \in \mathcal{S} \backslash \mathcal{B}_\mathsf{G}$ satisfying $(B \otimes I_3) e(R, x_c) = 0$, there exists $(i,j) \in \mathsf{E}'$ such that*

$$(R_i^\top R_j r_j, q_{ij}) \in (\mathcal{E}_{ij} \cup \mathcal{B}_{ij}) \backslash \mathcal{A}_{ij} \tag{24}$$

where

$$\mathcal{A}_{ij} := \{r_i\} \times \{r_i\} \tag{25a}$$
$$\mathcal{B}_{ij} := \{r_i\} \times Q_{ij} \tag{25b}$$
$$\mathcal{E}_{ij} := \{(x, q_{ij}) \in \mathbb{S}^2 \times Q_{ij} : x = \pm q_{ij}\} \tag{25c}$$

and Q_{ij} is given in (21), then $\mathcal{B}_\mathsf{G}$ in (19) is globally asymptotically stable for (23).

Proof. Let us consider the function

$$U(R, x_c) := \sum_{(i,j) \in L \backslash \mathsf{E}'} h(R_i^\top R_j r_j, r_i) + \sum_{(i,j) \in \mathsf{E}'} V_{ij}(R_i^\top R_j r_j, q_{ij})$$

for each $(R, x_c) \in \mathcal{S}$. Since V_{ij} is a synergistic potential function relative to r_i for each $(i,j) \in \mathsf{E}'$ by construction, then U is positive definite relative to

$$\mathcal{A}_\mathsf{G} := \{(R, x_c) \in \mathcal{S} : q_{ij} = r_i \text{ for all } (i,j) \in \mathsf{E}', R_i r_i = R_j r_j \text{ otherwise}\}$$

Letting C_G and D_G denote the fow set and the jump set of (23), respectively, the growth of V along solutions to (23) is globally bounded by u_c, u_d, where

$$u_c(R, x_c) := \begin{cases} -|(B \otimes I_3) e(R, x_c)|^2 & \text{if } (R, x_c) \in C_\mathsf{G} \\ -\infty & \text{otherwise} \end{cases}$$

$$u_d(R, x_c) := \begin{cases} -\delta & \text{if } (R, x_c) \in D_\mathsf{G} \\ -\infty & \text{otherwise} \end{cases}$$

for each $(R, x_c) \in \mathcal{S}$. The assumption that $B \otimes I_3) e(R, x_c) = 0$ implies (24), means that $(R, x_c) \notin C_\mathsf{G}$ for each $(R, x_c) \in \mathcal{S} \backslash \mathcal{B}_\mathsf{G}$ satisfying $(B \otimes I_3) e(R, x_c) = 0$. Therefore, $u_c(R, x_c), u_d(R, x_c) < 0$ for each $(R, x_c) \in \mathcal{S} \backslash \mathcal{A}_\mathsf{G}$, thus it follows from [20, Corollary 8.9] that $\mathcal{A}_\mathsf{G}$ is globally asymptotically stable for (23). The remainder of the proof follows from arguments similar to those in Lemma 5 and Theorem 1. □

For the particular case where G is a tree graph, we have that $\mathsf{G}' := (\mathsf{V}, L)$ induces a hybrid controller that globally asymptotically stabilizes the set $\mathcal{B}_\mathsf{G}$ for (23).

Lemma 8. *Given the network $\mathsf{G} := (\mathsf{V}, \mathsf{E})$ with incidence matrix B and the subgraph $\mathsf{G}' := (\mathsf{V}, L)$, if G is a tree graph, then, for each $(R, x_c) \in \mathcal{S} \backslash \mathcal{B}_\mathsf{G}$ satisfying $(B \otimes I_3) e(R, x_c) = 0$, there exists $(i,j) \in \mathsf{E}'$ such that (24) is satisfied.*

Proof. This result follows from the fact that, for a tree graph, the null space of $B \otimes I_3$ is equal to $\{0\}$ (c.f. [25]) □

VI. Simulation Results

In this section, we illustrate the behavior of the closed-loop hybrid system (23) for a network $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ of 3 vehicles, with $\mathsf{V} := \{1,2,3\}$ and $\mathsf{E} := \{\{1,2\},\{2,3\}\}$. We also consider the hybrid feedback links represented by the subgraph G' of G, given by $\mathsf{G}' := (\mathsf{V}', \mathsf{E}')$ with $\mathsf{V} := \{1,2,3\}$

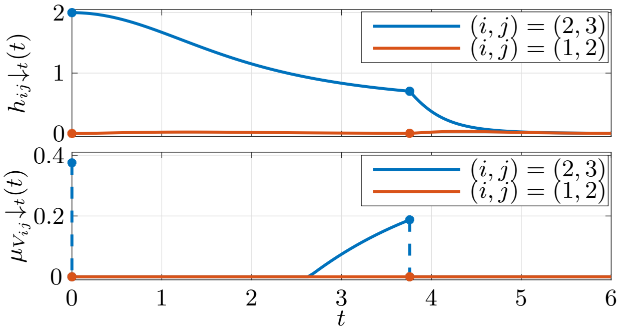

Fig. 1. (Top) Evolution of the distances between reference vectors of each agent with continuous time, given by $t \mapsto h_{ij}\downarrow_t(t) := 1 - r^\top R_i\downarrow_t(t)^\top R_j\downarrow_t(t) r$, for the initial conditions described in Section VI. (Bottom) Evolution of the synergy gap for each edge with continuous time for the initial conditions described in Section VI. Solid lines represent flows while dashed lines represent jumps.

and $\mathsf{E}' := \{(1,2),(2,3)\}$. It follows from Lemma 8 that the conditions of Theorem 2 are satisfied, hence $\mathcal{B}_{\mathsf{G}} = \{(R, x_c) \in \mathrm{SO}(3)^3 \times \mathcal{X} : R_i r_i = R_j r_j \text{ for all } i,j \in \{1,2,3\}\}$ with $x_c = (q_{12}, q_{23}) \in \mathcal{X} = (Q_r \cup \{r\})^2$, $r_i = r = [0\ 0\ -1]^\top$ for all $i \in \{1,2,3\}$ and $\gamma = 0.5$, is globally asymptotically stable for (23), using the synergistic hybrid feedback induced by V in (13) with parameters $\alpha = 0.875$, $\beta = 0.5$ and $\delta = 0.5 \min\{\delta_i : i \in \{1,2,3,4\}\}$, where $\delta_1 = 0.375$, $\delta_2 = 0.750$, $\delta_3 = 1.125$ and $\delta_4 = 0.375$ are obtained from (18).

Figure VI represents the evolution of the distance between the reference vectors of each agent and of the synergy gap with continuous time, starting from an initial condition where $R_1(0,0)r_1 = R_2(0,0)r_2 = -R_3(0,0)r_3$ and $q_{12}(0,0) = q_{23}(0,0) = r$. It is possible to verify that the distance between the reference vectors r of agents 2 and 3 is at a maximum initially, in accordance with the selected initial condition. This situation triggers a switch of the variable q_{23} which is identified in the bottom figure by the blue dashed line. As $R_2^\top R_3 r$ converges to q_{23} the synergy gap grows up to the point where a new switch is triggered at $t \approx 3.76$ and $q_{23} = r$ is selected. The final portion of the figure corresponds to the convergence of the state to $\mathcal{B}_{\mathsf{G}}$ as desired.

VII. Conclusions

The contributions in this paper were threefold: 1) We introduced a new concept of synergistic potential function on $\mathbb{S}^n$ and we show that it induces a controller that is able to globally asymptotically stabilize a given setpoint for the hybrid closed-loop system; 2) We constructed a synergistic potential function of this kind; 3) We showed that it can be applied to the problem of reduced attitude synchronization so that synchronization is achieved globally with respect to the initial conditions. We have provided simulation results that illustrate global reduced attitude synchronization for a network of three chained vehicles.

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