Improved Maneuverability for Multirotor Aerial Vehicles using Globally Stabilizing Feedbacks
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Abstract—In this paper, we present the design of trajectory tracking controllers for multirotor aerial vehicles that have the ability to operate both with and without thrust reversal. We follow a hierarchical control approach, in the sense that we start by designing a common saturated controller for the position subsystem and use it to provide a reference to an attitude tracking controller. The controllers for each operating mode are able to achieve global asymptotic stability as well as semiglobal exponential stabilization of the zero tracking error set. We demonstrate the capabilities of the proposed controllers in a simulation that performs a throw-and-catch maneuver.

I. INTRODUCTION

Nowadays, Uninhabited Aerial Vehicles (UAVs) can be bought in consumer electronic stores and piloted by the whisk of a smartphone. High power density batteries opened the skies to these remotely operated aerial vehicles and the development of new sensors, processors and actuators is fueling further innovation. A few of the areas of soaring research include: parcel delivery, video capture, entertainment, surveillance, infrastructure inspection and transportation (c.f. [1], [2], [3] and [4]). The class of UAVs known as multirotor aerial vehicles is one of the most popular solutions to these applications. These vehicles are equipped with several propellers that produce thrust and torque (see Figure 1). The simplicity of their mechanical design is in deep contrast with the control challenges that these poses. These difficulties lead to a decade long research effort described in [5] and [6] and that we summarize below.

The contributions in [7], [8] and [9] focus on the development and validation of dynamical models for multirotor aerial vehicles. In these papers, the controller has a PID structure with gains that are tuned for a limited region of the flight envelope. In this paper, we follow a control design approach that is more closely related to [10] and the line of research that originated from it. The main features of said line of research include: parcel delivery, video capture, entertainment, surveillance, infrastructure inspection and transportation. These vehicles are equipped with several propellers that produce thrust and torque (see Figure 1). Namely, we assume the concept of synergistic potential functions that was introduced in [18] and expand the contributions in [19] and [20] in the following ways: 1) we provide a smooth saturated feedback law for the position subsystem whose parameters can be selected to simultaneously achieve global asymptotic stabilization and semiglobal exponential stabilization of the zero tracking error set; 2) we demonstrate these stability properties can be extended to the full dynamical system by

1) Hierarchical Control Architecture: given a desired (smooth) position trajectory, the hierarchical control architecture consists of designing a virtual feedback law at the acceleration level and using it to define an attitude reference that is to be tracked by an inner attitude tracking controller.

2) Geometric Feedback: we say that the feedback law is geometric if it takes into account the topology of the space of the attitude representation unlike linearization approaches, which discard that information.

Some important contributions to multirotor control that share the aforementioned characteristics are given in [11], [12], [13] and [14]. The proposed approach differs from solutions that focus solely on attitude stabilization, such as the ones in [15] and [16]. For an in-depth comparison between multirotor control strategies, we refer the reader to [17].

Off-the-shelf multirotor vehicles usually have a number of different modes of operation. For example, the Blade 200qx quadrotor vehicle has the following operating modes: 1) position stabilization mode; 2) angle stabilization mode; 3) aerobatic mode. Depending on the operating mode, the internal controller maps Radio Frequency (RF) commands to position, angle or angular velocity commands. It is seldom the case that off-the-shelf multirotor vehicles allow for direct torque control, much less individual rotor control.

In this paper, we develop higher level controls for off-the-shelf multirotor vehicles that have similar functions to the Blade 200qx quadrotor of Figure 1. Namely, we assume that the input to the system is comprised of the thrust and angular velocity commands and we assume that the vehicle is capable of operating with or without thrust reversal. Using the concept of synergistic potential functions that was introduced in [18] we expand the contributions in [19] and [20] in the following ways: 1) we provide a smooth saturated feedback law for the position subsystem whose parameters can be selected to simultaneously achieve global asymptotic stabilization and semiglobal exponential stabilization of the zero tracking error set; 2) we demonstrate these stability properties can be extended to the full dynamical system by
means of synergistic hybrid feedback. In particular, we use
different synergistic potential functions when thrust reversal
is available and when it is not, which lead to very different
behavior of the closed loop system. The behavior of the
closed-loop systems is illustrated with simulations of throw-
and-catch maneuvers.

This paper is organized as follows: Section II presents
the notation and mathematical concepts that are used throughout
the paper; Section III formally introduces the problem at
hand; Section IV presents the design of the controller for
the position subsystem; Section V completes the controller
design for the full system using synergistic hybrid feedback
with and without thrust reversal; Section VI presents the
conclusions of the paper. The proofs of the results in this
paper were omitted due to space constraints, but will appear
elsewhere.

II. PRELIMINARIES AND NOTATION

\( \mathbb{N} \) denotes the natural numbers and 0. Given \( n, m \in \mathbb{N} \):
\( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space with norm
\( |x| := \sqrt{x^T x} \) for each \( x \in \mathbb{R}^n \); \( I_n \in \mathbb{R}^n \) denotes an
\( n \)-dimensional vector of ones; \( 0_n \in \mathbb{R}^n \) denotes the \( n \)-
dimensional zero vector; \( e_i \in \mathbb{R}^n \) is defined for each \( i \in \{1, 2, \ldots, n\} \) and represents a vector whose entries are all
zero except for the \( i \)-th component which is equal to 1; \( \mathbb{R}^{m \times n} \)
denotes the space of \( m \times n \) matrices; \( I_{m \times n} \) denotes the \( n \times n \) identity matrix \( 0_{m \times n} \) denotes an \( m \times n \) matrix of zeros;
the special orthogonal group is \( \text{SO}(n) := \{ R \in \mathbb{R}^{n \times n} : R^T R = I_n, \det(R) = 1 \} \); the \( n \)-dimensional sphere is represented by
\( S^n := \{ x \in \mathbb{R}^{n+1} : x^T x = 1 \} \); a set-valued mapping
\( M \) from \( \mathbb{R}^m \) to \( \mathbb{R}^n \) is denoted by \( M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n \) and it is a function that maps vectors in \( x \in \mathbb{R}^m \) to subsets of \( \mathbb{R}^n \), i.e., \( M(x) \subset \mathbb{R}^n \) for each \( x \in \mathbb{R}^m \); the closed
\( n \)-dimensional ball centered at \( c \) with radius \( \gamma \) is denoted by
\( c + \gamma B := \{ x \in \mathbb{R}^n : |x - c| \leq \gamma \} \). Given a set
\( S \subset \mathbb{R}^n \), we define \( S + \gamma B := \bigcup_{s \in S} (s + \gamma B) \). Given \( v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n \), the mapping \( v \rightarrow \text{diag}(v) \) represents the
diagonal matrix whose diagonal entries are the components of \( v \). If \( v_i < 0 \) for each \( i \in \{1, 2, \ldots, n\} \), then we write \( v \prec 0_n \). Given a set-valued mapping \( M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n \), we define \( \text{rge} M := \{ y \in \mathbb{R}^n : \exists x \in M \text{ such that } y = M(x) \} \).

Let \( P \in \mathbb{R}^{n \times n} \). if \( P = P^T \), then \( P \) is said to be symmetric.

Real symmetric matrices have real eigenvalues and we denote
the highest and lowest eigenvalue of \( P \) by \( \lambda_{\max}(P) \) and
\( \lambda_{\min}(P) \), respectively. A symmetric matrix \( P \in \mathbb{R}^{n \times n} \) is said to be positive definite if all of its eigenvalues are positive,
in which case we write \( P > 0_{n \times n} \). A symmetric matrix \( P \) is said to be positive semidefinite if all of its eigenvalues are nonnegative, in which case we write \( P \succeq 0_{n \times n} \). Given
\( A \in \mathbb{R}^{m \times n} \), \( \text{vec}(A) = \begin{bmatrix} e_1^T A^T & e_2^T A^T & \cdots & e_n^T A^T \end{bmatrix}^T \). Given a continuously differentiable function \( F : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{p \times q} \), \( DF(X) := \frac{\partial \text{vec}(F)}{\partial \text{vec}(X)}(X) \) for each \( X \in \mathbb{R}^{m \times n} \). A
function \( V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) is said to be positive definite relative
\( \mathbb{R}^n \) to \( A \in \mathbb{R}^{n \times n} \) if it is positive definite relative to \( \mathbb{R}^n \) and there exists \( \delta > 0 \) such that
\[
\mu_V(x, q) := V(x, q) - \min \{ V(x, q) : q \in \mathbb{Q} \} > \delta \quad (1)
\]
for each \( (x, q) \in \mathbb{E}_V \setminus \mathbb{A} \), with \( \mathbb{E}_V := \{ (x, q) \in \mathbb{S}^n \times \mathbb{Q} : \Pi(x) \nabla V(x, q) = 0_{n+1} \} \) and \( \Pi(x) := I_{n+1} - xx^T \) for each
\( x \in \mathbb{S}^n \) and \( \nabla V(x, q) := \frac{\partial V}{\partial q}(x, q) \) for each \( (x, q) \in \mathbb{S}^n \times \mathbb{Q} \).
In addition, if (1) is verified, we also say that \( V \) has synergy
gap exceeding \( \delta \).

Given a synergistic potential function on \( \mathbb{S}^n \) relative to
\( \mathbb{S}^n \times \mathbb{Q} \) with synergy gap exceeding \( \delta \), we define:
\[
C_V := \{ (x, q) \in \mathbb{S}^n \times \mathbb{Q} : \mu_V(x, q) \leq \delta \}, \quad (2a)
\]
\[
D_V := \{ (x, q) \in \mathbb{S}^n \times \mathbb{Q} : \mu_V(x, q) \geq \delta \}. \quad (2b)
\]

III. PROBLEM SETUP

The dynamics of a thrust vectored vehicle can be described by
\[
\dot{p} = v \quad (3a)
\]
\[
m\ddot{v} = RrT + mg \quad (3b)
\]
\[
\dot{R} = RS(\omega) \quad (3c)
\]
where \( S(\omega) \) is such that \( S(\omega)v = \omega \times v \) for all \( \omega, v \in \mathbb{R}^3 \);
\( p \in \mathbb{R}^2 \) and \( v \in \mathbb{R}^3 \) denote the position and the velocity of
the vehicle with respect to the inertial reference frame (in inertial
coordinates), \( R \in \text{SO}(3) \) is the rotation matrix that maps
vectors in body-fixed coordinates to inertial coordinates, \( g \in \mathbb{R}^3 \)
represents the gravity vector and \( r \in \mathbb{S}^2 \) is the thrust
vector in body-fixed coordinates. Furthermore, the
inputs to (3) are \( \omega \in \mathbb{R}^3 \) and \( T \in \mathbb{R} \) which
represent the angular velocity in body-fixed coordinates and the
magnitude of the thrust, respectively. The dynamical model (3) is a
simplification of the one provided in [10] that better suits
our experimental setup. Suppose that we are given a reference
trajectory satisfying the following assumption.

Assumption 1. The reference trajectory \( t \mapsto p_d(t) \) is a thrice
continuously differentiable path defined for each \( t \geq 0 \) that
satisfies \( \text{rge} p_d^{(3)} \subset \mathcal{R}_3 \) for some compact and convex set
\( \mathcal{R}_3 \subset \mathbb{R}^3 \) and \( \text{rge} \dot{p}_d \subset \mathcal{R}_2 \subset \mathbb{R}^3 \).

Given a reference trajectory that satisfies Assumption 1, we
define the tracking errors
\[
e_p := p - p_d \quad (4a)
e_v := v - \dot{p}_d \quad (4b)
\]
whose dynamics can be derived from (3) and are given by:
\[
\dot{e}_p = e_v, \quad \dot{e}_v = \frac{RrT}{m} + g - \dot{p}_d. \quad (5)
\]
If we assume that we have full control over \( R \in \text{SO}(3) \) and
\( T \in \mathbb{R} \), then it is possible to reduce the trajectory tracking
problem to the problem of stabilizing a double integrator.
This is the approach that we follow in the next section.

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IV. Controller for the Position Subsystem

Let \( \ell := (\ell_1, \ell_2, \ell_3) \in \mathbb{R}^3 \) denote a vector with positive components and let \( \sigma(u) := [\sigma_1(u_1) \ \sigma_2(u_2) \ \sigma_3(u_3)]^\top \) represent a function that is defined for each \( u := (u_1, u_2, u_3) \in \mathbb{R}^3 \), with \( \sigma_i \) continuously differentiable, strictly increasing and satisfying
\[
\sigma_i(u_i) = u_i \quad \forall u_i \in [-\ell_i, \ell_i] \tag{6}
\]
for each \( i \in \{1, 2, 3\} \).

**Example 1.** Given \( \ell := (\ell_1, \ell_2, \ell_3) \in \mathbb{R}^3 \) and \( M := (M_1, M_2, M_3) \in \mathbb{R}^3 \) satisfying \( \ell_i < M_i \), \( \forall i \in \{1, 2, 3\} \), the function
\[
\sigma_i(u_i) := \begin{cases} \\
-\ell_i + \frac{2(M_i - \ell_i)}{\pi} \arctan \left( \frac{\pi(u_i + \ell_i)}{2(M_i - \ell_i)} \right) \quad &\text{if } u_i < -\ell_i \\
u_i &\text{if } |u_i| \leq \ell_i \\
\ell_i + \frac{2(M_i - \ell_i)}{\pi} \arctan \left( \frac{\pi(u_i - \ell_i)}{2(M_i - \ell_i)} \right) \quad &\text{if } u_i > \ell_i 
\end{cases}
\]
is defined for each \( u_i \in \mathbb{R} \) and it is strictly increasing, continuously differentiable and satisfies (6).

The function \( \sigma \) can be used to generate a reference for the thrust vector as follows
\[
r_d(p_d, e) := \frac{\eta(p_d, e)}{\eta(p_d, e)} \tag{7}
\]
with \( e := (e_p, e_v) \in \mathbb{R}^6 \),
\[
\eta(p_d, e) = \sigma(Ke) + g - p_d
\]
for each \( (p_d, e) \in \mathbb{R}_+ \times \mathbb{R}^6 \) such that \( \eta(p_d, e) \neq 0 \) and \( K \in \mathbb{R}^{3 \times 6} \). The following assumption guarantees that \( r_d \) in (7) is always well-defined.

**Assumption 2.** Given \( g \in \mathbb{R}^3 \) as in (3) and a reference trajectory satisfying Assumption 1, the following holds
\[
\sigma(u) + g - p_d \neq 0 \quad \forall (u, p_d) \in \mathbb{R}^3 \times \mathbb{R}_+
\]
\[
\text{If } RrT/m \equiv \eta(p_d, e), \text{ then (5) can be written as}
\]
\[
\ddot{e} = Ae + B\sigma(Ke), \tag{9}
\]
with
\[
A := \begin{bmatrix} 0_{3\times 3} & I_3 \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}, \quad B := \begin{bmatrix} 0_{3\times 3} \\ I_3 \end{bmatrix}.
\]

Using \( e_p := (e_{p,1}, e_{p,2}, e_{p,3}) \) and \( e_v := (e_{v,1}, e_{v,2}, e_{v,3}) \) and
\[
K = \begin{bmatrix} k_{1,1} & 0 & 0 & k_{1,2} & 0 & 0 \\ 0 & k_{2,1} & 0 & 0 & k_{2,2} & 0 \\ 0 & 0 & k_{3,1} & 0 & 0 & k_{3,2} \end{bmatrix}, \tag{10}
\]
with \( k_i := (k_{i,1}, k_{i,2}) \prec \ell_2 \) for each \( i \in \{1, 2, 3\} \), the system (9) becomes a collection of three parallel double integrators, each of which is characterized by the dynamics \( \dot{e}_i = (e_{v,i}, \sigma_i(k_i e_i)) \) where \( e_i := (e_{p,i}, e_{v,i}) \in \mathbb{R}^2 \) for each \( i \in \{1, 2, 3\} \). In the next lemma, we present a Lyapunov function candidate for each double integrator that has a number of properties that are used in the sequel.

**Lemma 1.** For each \( i \in \{1, 2, 3\} \), \( \ell_{p,i} > 0 \), \( \ell_i > 0 \) and \( k_i < \ell_i \), there exists \( P_i > 0_{2 \times 2} \) such that:

**(P1)** The function \( V_{p,i} \), defined for each \( e_i \in \mathbb{R}^2 \), as follows:
\[
V_{p,i}(e) := \frac{e_i}{2} \left[ \begin{array}{c} \sigma_i(k_i e_i) \\ e_{v,i} \end{array} \right]^\top P_i \left[ \begin{array}{c} \sigma_i(k_i e_i) \\ e_{v,i} \end{array} \right] + e_i \int_0^{k_i e_i} \sigma(\tau) d\tau
\]
with \( e_i = \frac{2\ell_{p,i}}{\ell_i^2} \)

is continuously differentiable, positive definite and radially unbounded.

**(P2)** There exists a positive definite function \( W_{p,i} : \mathbb{R}^2 \to \mathbb{R}_{\geq 0} \) such that
\[
DV_{p,i}(e_i) \left[ \begin{array}{c} e_{v,i} \\ \sigma_i(k_i e_i) \end{array} \right] \leq -W_{p,i}(e_i)
\]
for each \( e_i \in \mathbb{R}^2 \).

**(P3)** Each \( e_i \)
\[
\Omega_{V_{p,i}}(e_i) := \{ e_i \in \mathbb{R}^2 : V_{p,i}(e_i) \leq \ell_{p,i} \}
\]
satisfies \( k_i e_i \leq \ell_i \).

**(P4)** There exists \( \mu_i > 0 \) such that
\[
DV_{p,i}(e_i) \left[ \begin{array}{c} e_{v,i} \\ \sigma_i(k_i e_i) \end{array} \right] \leq -\mu_i V_{p,i}(e_i)
\]
for each \( e_i \in \mathbb{R}^2 \) satisfying \( k_i e_i \leq \ell_i \).

**(P5)** There exists \( \alpha_i > 0 \) such that \( e_i e_i^2 \leq V_{p,i}(e_i) \leq \alpha_i e_i^2 \) for all \( e_i \in \mathbb{R}^2 \) satisfying \( k_i e_i \leq \ell_i \).

**(P6)** There exists a continuous function \( \rho_i : \mathbb{R}^2 \to \mathbb{R}_{\geq 0} \) such that
\[
\frac{DV_{p,i}(e_i)}{e_{v,i}} (V_{p,i}(e_i))^{-\frac{1}{2}} \leq \rho_i(e_i)
\]
for each \( e_i \in \mathbb{R}^2 \).

Properties (P1) and (P2) can be used to prove that the origin of (9) is globally asymptotically stable. Together with property (P3), we have that solutions to (9) starting in the sublevel set (11) do not exceed the bounds \( \ell_i \) in (6).

Properties (P4) and (P5) allow for semiglobal exponential stability, as shown in the following sections.

V. Global Asymptotic Tracking

A. With Thrust Reversal

We start the controller design with the construction of a synergistic potential function on \( \mathbb{S}^2 \).

**Lemma 2.** Let \( \delta_i \in (0, 2) \), \( Q_1 := \{ -1, 1 \} \) and \( A_1 := \{(x, q) \in \mathbb{S}^2 \times Q_1 : x = q r\} \) with \( r \in \mathbb{S}^2 \) given in (3). The function \( V_1(x, q) := 1 - q^2 r^2 \) is a synergistic potential function on \( \mathbb{S}^2 \) relative to \( A_1 \) with synergy gap exceeding \( \delta_1 \). Moreover, there exists \( \lambda_1 > 0 \) such that
\[
\|\Pi(x) \nabla V_1(x, q)\|^2 \geq \lambda_1 V_1(x, q)
\]
for each \( x \in C_1 \).

Let \( z_1 := (\tilde{p}_d, e, R, q_1) \in Z_1 := \mathbb{R}_+ \times \mathbb{R}^6 \times SO(3) \times Q_1 \). We define the hybrid controller
\[
\dot{q}_1 = 0 \quad z \in C_1 := \{ z_1 \in Z_1 : (R^T r_d(\tilde{p}_d, e), q_1) \in C_V \}
\]
\[
q_1^+ = -q_1 \quad z \in D_1 := \{ z_1 \in Z_1 : (R^T r_d(\tilde{p}_d, e), q_1) \in D_V \}.
\]

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where $\mu V_1$ is given in (1) and $C V_1$, $D V_1$ are given in (2). Given $\sigma$ as in Section III, $K \in \mathbb{R}^{3 \times 6}$ as in (10) and
\[
V_p(e) := \sum_{i=1}^{3} V_{p,i}(e_i)
\]
for each $e \in \mathbb{R}^6$, we assign the input $T$ of (3) to
\[
T_1(p_{d}, e, R) = r^T \mathbf{R}^T \eta(p_{d}, e) \nabla S(p_{d}, e) + \mathbf{R}^T \eta_1(p_{d}(3), p_{d}, e) \frac{\mathbf{R}_T r_d(p_{d}, e)}{\alpha} + \left( S(r^T r_d(p_{d}, e)) \right) \nabla (\eta(p_{d}, e)) \nabla S(p_{d}, e) + \mathbf{R}^T \eta_1(p_{d}(3), p_{d}, e) \frac{\mathbf{R}_T r_d(p_{d}, e)}{\alpha} + \left( S(r^T r_d(p_{d}, e)) \right) \nabla (\eta(p_{d}, e)) \nabla S(p_{d}, e)
\]
with $\eta(p_{d}, e)$ given in (8), hence $T$ can take both positive as well as negative values. Moreover, we assign the input $\omega$ of (3) to
\[
\omega_1(p_{d}(3), z_1) = -S(r)^2 \left( q_1 \beta_1 S(r) R^T q_d + \right) \frac{m^{-1} R^T r_d(p_{d}, e)}{\alpha} + \frac{1}{\alpha} \left( S(r^T r_d(p_{d}, e)) \right) \nabla (\eta(p_{d}, e)) \nabla S(p_{d}, e)
\]
with $\alpha, \beta_1 > 0$,
\[
\eta_1(p_{d}(3), p_{d}, e) \equiv \mathbf{D} \eta(p_{d}, e) \left[ \begin{array}{c} p_{d} \\ R r T_1(p_{d}, e) \\ m \end{array} \right] + g - p_{d}
\]
and $\nu(p_{d}, e, R) := \left( \eta(p_{d}, e) \right) S(r) \mathbf{R}_d^T \mathbf{D} V_1(e) \nabla S(p_{d}, e) \frac{\mathbf{R}_T r_d(p_{d}, e)}{\alpha} + \left( S(r^T r_d(p_{d}, e)) \right) \nabla (\eta(p_{d}, e)) \nabla S(p_{d}, e)$ for each $(p_{d}, e, R) \in \mathbb{R}_2 \times \mathbb{R}^6 \times SO(3)$ (13). The closed-loop system is given by the hybrid system $H_1 := (C_1, F_1, D_1, G_1)$ with
\[
F_1(z_1) := \left\{ \begin{array}{c} \left( \begin{array}{c} p_{d}(3) \\ R r T_1(p_{d}, e, R) \\ m \end{array} \right) + g - p_{d} \\ \nabla (\eta_1(p_{d}(3), p_{d}, e)) \frac{\mathbf{R}_T r_d(p_{d}, e)}{\alpha} + \left( S(r^T r_d(p_{d}, e)) \right) \nabla (\eta(p_{d}, e)) \nabla S(p_{d}, e) \end{array} \right\} : p_{d} \in \mathbb{R}_3
\]
for each $z_1 \in C_1$ and $G_1(z_1) := (p_{d}, e, R, -q)$ for each $z_1 \in D_1$. The next result asserts the global asymptotic stability of the zero tracking error set for the closed-loop system $H_1$. We refer the reader to a similar proof in [22].

**Proposition 1.** Let Assumptions 1 and 2 hold and let $K$ be given by (10). The set
\[
B_1 := \{ z_1 \in \mathbb{Z} : e = 0, (R^T r_d(p_{d}, e), q_1) \in \mathcal{A}_1 \}
\]
is globally asymptotically stable for $H_1$.

In addition to global asymptotic stability of $B_1$ for $H_1$, we also show that $B_1$ is: 1) locally exponentially stable for $H_1$, because there exists a neighborhood of $B_1$ from which solutions converge exponentially to $B_1$; 2) semiglobally exponentially stable for $H_1$, because, for each compact set of initial conditions $(e, R)(0, 0) \in \Omega \subset \mathbb{R}^6 \times SO(3)$, there exists a controller gain $K$ that guarantees exponential convergence to $B_1$ for solutions starting in that set.

**Proposition 2.** Let Assumptions 1 and 2 hold and let $K$ be given by (10). There exist $c, \lambda, \gamma > 0$ such that, for each solution $\phi$ to $H_1$ with initial condition $\phi(0, 0) \in (B_1 + \gamma B) \cap \mathbb{Z}_1$, the following holds
\[
|\phi(t, j)|_{B_1} \leq c |\phi(0, 0)|_{B_1} \exp(-\lambda t) \quad (14)
\]
for all $(t, j) \in \text{dom } \phi$.

**Proposition 3.** Let Assumptions 1 and 2 hold and let $K$ be given by (10). For each compact set $\Omega \subset \mathbb{R}^6 \times SO(3)$ there exist $c, \lambda > 0$ and $k_i \in \mathbb{R}^2$ for all $i \in \{1, 2, 3\}$ such that, for each solution $\phi$ to $H_1$ with initial condition $\phi(0, 0) \in \mathbb{R}_2 \times \Omega \times \mathbb{Q}_1$, (14) holds for all $(t, j) \in \text{dom } \phi$.

Semiglobal exponential stability is achieved in Proposition 3 through a low gain design, meaning that it is possible to encompass larger and larger compact sets of initial conditions choosing smaller and smaller controller gains $K$, but this has adverse effects on the convergence rate of the position subsystem. In the next section, we design a controller for the case without thrust reversal.

**B. Without Thrust Reversal**

We start with the assumption that there exists a synergistic potential function relative to $r$.

**Assumption 3.** Given a compact set $Q_2$ and $A_2 := \{ r \times Q_2 \}$ with $r \in S^2$ as in (3), there exists a synergistic potential function $V_2 : S^2 \times Q_2 \rightarrow \mathbb{R}_2 \times \Omega \times \mathbb{Q}_1$ relative to $A_2$ with synergy gap exceeding $\delta$ satisfying
\[
\begin{align*}
&c_1 \|(x, q_2)\|_{A_2} \leq V_2(x, q) \leq c_2 \|(x, q_2)\|_{A_2} \quad \forall (x, q_2) \in \mathbb{R}_2 \\
&\|\Pi(x) V_2(x, q_2)\|_{A_2} \geq c_3 V_2(x, q_2) \quad \forall (x, q_2) \in C_{V_2}
\end{align*}
\]
for some $c_1, c_2, c_3 > 0$, where $C_{V_2}$ is given in (2a).

**Example 2.** It is shown in [18] that the function
\[
V_2(x, q_2) := \frac{1 - r^T x}{2 - r^T x - q_2^2 x} \quad \forall (x, q_2) \in \mathbb{S}^2 \times \mathbb{Q}_2
\]
with $\mathbb{Q}_2 := \{ q_2 \in \mathbb{S}^2 : q_2^T r \leq 0 \}$ satisfies Assumption 3.

Under Assumption 3, let $z_2 := (p_d, e, R, q_2) \in \mathbb{Z}_2 := \mathbb{R}_2 \times \mathbb{R}^6 \times SO(3) \times \mathbb{Q}_2$ and the hybrid controller
\[
\begin{align*}
\dot{q}_2 &= 0 \\
q_2^T &= \rho(g(R^T r_d(p_{d}, e))) \quad z_2 \in D_2,
\end{align*}
\]
with
\[
\begin{align*}
&z_2 \in C_2 := \{ z_2 \in \mathbb{Z}_2 : (R^T r_d(p_{d}, e), q_2) \in C_{V_2} \} \\
z_2 \in D_2 := \{ z_2 \in \mathbb{Z}_2 : (R^T r_d(p_{d}, e), q_2) \in D_{V_2} \}
\end{align*}
\]
and $\rho(x) := \arg \min \{ V_2(x, q) : q \in \mathbb{Q}_2 \}$ for each $x \in \mathbb{S}^2$. Let $\alpha > 0$ and $\nu_2$ denote a continuous function satisfying
\[
\alpha \sqrt{c_3} \nu_2(p_{d}, e) \geq |\eta(p_{d}, e)| \rho(e)
\]
for each $(p_{d}, e) \in \mathbb{R}_2 \times \mathbb{R}^6$ with
\[
\rho(e) := \sum_{i=1}^{3} \rho_i(e_i) \quad \forall e \in \mathbb{R}^6
\]
and $\rho_i$ given in Lemma 1. Given $\eta$ in (8), we assign the input $T$ in (3) to
\[
T_2(p_{d}, e) = m \eta(p_{d}, e) \quad \forall (p_{d}, e) \in \mathbb{R}_2 \times \mathbb{R}^6
\]
for all $(t, j) \in \text{dom } \phi$.
and the input $\omega$ in (3) to
$$\omega_2(p^{(3)}_d, z) = S(R^T r_d(\ddot{p}_d, e)) (R^T \eta_2(p^{(3)}_d, z_2)) + (\beta_2 + \nu_2(p^{(3)}_d, e)) \nabla V_2(R^T r_d(p^{(3)}_d, e), q_2)$$

for each $(p^{(3)}_d, z_2) \in \mathcal{R}_3 \times \mathcal{Z}_2$, with $\beta_2 > 0$ and
$$\eta_2(p^{(3)}_d, z_2) \equiv D\eta(p^{(3)}_d, e) \left[ p^{(3)}_d - \frac{R r^T T_2(p^{(3)}_d, e, R)}{m} \right] + g - \ddot{p}_d.$$ 

The closed-loop system is given by the hybrid system $\mathcal{H}_2 := (C_2, F_2, D_2, G_2)$ with data
$$F_2(z_2) := \left\{ \begin{array}{l} p^{(3)}_d \\ R r^T T_2(p^{(3)}_d, e, R) + g - \ddot{p}_d \\ R S(\omega_2(p^{(3)}_d, z_2)) \end{array} : p^{(3)}_d \in \mathcal{R}_3 \right\}$$
for each $z_2 \in C_2$ and $G_2(z_2) := (\ddot{p}_d, e, R, \dot{g}(R^T r_d(p^{(3)}_d, e)))$ for each $z_2 \in D_2$. The following proposition shows that the zero tracking error set is globally asymptotically stable for the closed-loop system $\mathcal{H}_2$.

**Proposition 4.** Let Assumptions 1 and 2 hold and let $K$ be given by (10). For each $k_i < 0$ and each $i \in \{1, 2, 3\}$, the set
$$B_2 := \{ z_2 \in \mathcal{Z}_2 : e = 0, (R^T r_d(\ddot{p}_d, e), q_2) \in A_2 \}$$
is globally asymptotically stable for $\mathcal{H}_2$.

**Proposition 5.** Let Assumptions 1 and 2 hold and let $K$ be given by (10). For each $k_i < 0$ and each $i \in \{1, 2, 3\}$, there exist $c, \lambda, \gamma > 0$ such that, for each solution $\phi$ to $\mathcal{H}_2$ with initial condition $\phi(0, 0) \in (B_2 + \gamma \mathcal{B}) \cap \mathcal{Z}_2$, the following holds
$$|\phi(t, j)|_{\mathcal{B}_2} \leq c |\phi(0, 0)|_{\mathcal{B}_2} \exp(-\lambda t)$$
for all $(t, j) \in \text{dom} \, \phi$.

**Proposition 6.** Let Assumptions 1 and 2 hold and let $K$ be given by (10). For each compact set $\Omega \subset \mathbb{R}^q \times SO(3)$ there exist $k_i \in \mathbb{R}^2$ for all $i \in \{1, 2, 3\}$ and $c, \lambda > 0$ such that, for each solution $\phi$ to $\mathcal{H}_2$ with initial condition $\phi(0, 0) \in \mathcal{R}_2 \times \Omega \times \mathcal{Q}_2$, the (18) holds for all $(t, j) \in \text{dom} \, \phi$.

### VI. Simulation Results

In this section, we present simulation results for the closed-loop systems that result from the interconnection between the dynamical system (3) and the controllers that are presented in Sections (V-A) and V-B. In these simulations, we consider that the acceleration of gravity is $g := (0.0, 9.81) \text{ m/s}^2$ and that the thrust vector is given by $r := (0, 0, -1)$ in body-fixed coordinates. For controller design purposes, let us consider that the maximum acceleration and jerk are $(2\pi f)^2 r_0$ and $(2\pi f)^3 r_0$, respectively, corresponding to the acceleration and jerk of a circular trajectory of radius $r_0 > 0$ and rotations per second $f \in \mathbb{R}$. Assumption 1 is satisfied with $\mathcal{R}_2 = \{ \ddot{p}_d \in \mathbb{R}^3 : |\ddot{p}_d| \leq (2\pi f)^2 r_0 \}$ and $\mathcal{R}_3 = \{ \ddot{p}_d \in \mathbb{R}^3 : |\ddot{p}_d| \leq (2\pi f)^3 r_0 \}$. Using the function $\sigma$ in Example 1, we have that $|\sigma(u)| < |M|$ for each $u \in \mathbb{R}^3$, thus the bound
$$|T_i(\ddot{p}_d, e)| < m(|M| + |g| + (2\pi f)^2 r_0)$$
holds for the given reference trajectory and for all $i \in \{1, 2\}$, corresponding to the thrust commands (13) and V-B, respectively. Given that the quadrotor in our experimental setup weighs $m = 0.216 \text{ kg}$ and its thrust is limited to $T_{max} = 1.3 m |g|$, it follows from (19) that $M$, $r_0$, and $f$ must satisfy $|M| + (2\pi f)^2 r_0 < 0.3 |g|$ for the control signal to be within the actuator constraints. To strike a good balance between trajectory tracking and compensation of position/velocity tracking errors we restrict the analysis to that immediately after the half flip, there is a time period the attitude control gains are on at $x$ the body-fixed and each

$$q \equiv Dq \left( \frac{R r^T T_2(p^{(3)}_d, e, R)}{m} \right) + g - \ddot{p}_d.$$ 

We select the controller gain $K$ as the solution to the LQR problem considering $R_{LQR} := I_3$ and $Q_{LQR} := \text{diag}([1 \, 0.01 \, 1])$. More specifically, $K := -R_{LQR}^{-1}P_{LQR}$, where $P_{LQR} > 0_{6 \times 6}$ is the solution to the Algebraic Riccati Equation
$$A^T P_{LQR} + P_{LQR} A - P_{LQR} R_{LQR}^{-1} P_{LQR}^T + Q_{LQR} = 0.$$ 

In order to test the controllers of Sections V-A and V-B, we carry out an input driven throw-and-catch maneuver for both closed-loop systems by means of a simulation that takes into consideration the dynamics of $\omega$, given by $\omega = 12\pi (\omega; p^{(3)}_d, z_2) - \omega$ for each $i \in \{1, 2\}$. More specifically, we consider that the position reference is the origin of the inertial reference frame and that there is an open-loop command $\omega_d(t) = (18.5, 0, 0)$ rad/s for each $t \in [1, 1 + \pi/18.5]$ corresponding to a half flip around the body-fixed $x$-axis during which the trajectory tracking controller is switched off. The controllers are switched back on at $t \geq 1 + \pi/18.5$ s and we analyze the capabilities of each closed-loop system to stabilize the desired setpoint. The initial condition is $p(0, 0) = 0_3$, $v(0, 0) = 0_3$, $R(0, 0) = I_3$, the attitude control gains are $a = 100$ and $\beta_2 = 1$, and the function $V_2$ is given in Example 2. The initial values of the logical variables $q_1$ and $q_2$ are 1 and $-r$, respectively.

Figure 2 represents the attitude tracking error and the angular velocity from $t = 0.8$ s up to $t = 2$ s. The open-loop command $\omega_d(t)$ starts at $t = 1$ s and ends roughly at $t = 1.3$ s, as represented by the shaded region. It is possible to verify that: 1) the behavior of the closed-loop systems is indistinguishable up until the end of the half flip; 2) at the end of the half flip, the controller from Section V-A performs position stabilization in inverted flight while the controller from Section V-B has to perform another half flip to return to upright flight condition. Figure 3 represents the thrust commands for both controllers and it is possible to check that both signals are identical before the throw maneuver, but converge to symmetric values after the half flip. Note that immediately after the half flip, there is a time period in which the controller with no thrust reversal accelerates towards the ground leading to increased position tracking errors as represented in Figure 4.
Fig. 2. Simulation results depicting the evolution over time of the distance of the thrust vector to the reference and of the angular velocity of the vehicle for a half flip. The shaded region spans the time in which open-loop command $\omega_d(t)$ is active.

Fig. 3. Simulation results depicting the evolution over time of the commanded thrust. The shaded region spans the time in which open-loop command $\omega_d(t)$ is active.

VII. CONCLUSIONS

In this paper, we provided trajectory tracking controllers for multirotor aerial vehicles that are capable of operating with or without thrust reversal. We have shown that it is possible to attain global asymptotic stabilization as well as semiglobal exponential stabilization of the error zero set in both operating modes. We compared the behaviour of the closed-loop systems by means of the simulation of a throw-and-catch maneuver, namely their ability to recover hovering flight from the disadvantaged conditions at the end of the maneuver. Further research on this topic will focus on the validation of the proposed controllers by means of experimental results.

REFERENCES


