Automatica 110 (2019) 108534

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

# Robust global exponential stabilization on the $n$-dimensional sphere with applications to trajectory tracking for quadrotors☆

Pedro Casau [a,1,*], Christopher G. Mayhew, Ricardo G. Sanfelice [b], Carlos Silvestre [a,2]

[a] Department of Electrical and Computer Engineering, Faculty of Science and Technology, University of Macau, Taipa, Macau, China
[b] Department of Electrical and Computer Engineering, University of California, Santa Cruz, CA 95064, USA

ARTICLE INFO

*Article history:*
Received 11 July 2017
Received in revised form 5 June 2019
Accepted 25 July 2019
Available online 20 September 2019

*Keywords:*
Hybrid control systems
Synergistic potential functions
Lyapunov-based control

ABSTRACT

In this paper, we design a hybrid controller that globally exponentially stabilizes a system evolving on the $n$-dimensional sphere, denoted by $\mathbb{S}^n$. This hybrid controller is induced by a "synergistic" collection of potential functions on $\mathbb{S}^n$. We propose a particular construction of this class of functions that generates flows along geodesics of the sphere, providing convergence to the desired reference with minimal path length. We show that the proposed strategy is suitable to the exponential stabilization of a quadrotor vehicle.



## 1. Introduction

### 1.1. Motivation and problem statement

In this paper, we design a hybrid controller for global exponential stabilization of a setpoint for a system evolving on the $n$-dimensional sphere, given by $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : x^\top x = 1\}$. The dynamics of this system can be described by

$$\dot{x} = \Pi(x)\omega \quad x \in \mathbb{S}^n, \tag{1}$$

where $\omega \in \mathbb{R}^{n+1}$ is the input and $\Pi(x) = I_n - xx^\top$ projects $\omega$ onto the tangent space to $\mathbb{S}^n$ at $x$, given by $\mathsf{T}_x\mathbb{S}^n := \{z \in \mathbb{R}^{n+1} : z^\top x = 0\}$. Even though there exist controllers that globally asymptotically stabilize a setpoint on the $n$-dimensional sphere (cf. Mayhew and Teel (2013a)) and others that globally exponentially stabilize a setpoint on the special orthogonal group of order 3 (cf. Berkane, Abdessameud, and Tayebi (2017) and Lee (2015)), to the best of our knowledge, the problem of global exponential stabilization of a setpoint in $\mathbb{S}^n$ has not been addressed before, despite the fact that it has many meaningful applications, such as visual servoing (cf. Triantafyllou, Rovithakis, and Doulgeri (2018)); control of robotic manipulators (cf. Chaturvedi and McClamroch (2009)); exoskeleton tracking (cf. Brahmi, Saad, Rahman, and Ochoa-Luna (2019)); multi-agent synchronization (cf. Markdahl, Thunberg, and Gonçalves (2018)); formation control (cf. Zhao and Zelazo (2016)); rigid-body stabilization (cf. Chaturvedi, Sanyal, and McClamroch (2011)) and trajectory tracking for multi-rotor aerial vehicles (cf. Mahony, Kumar, and Corke (2012)), which we also explore in this paper. To see how the proposed controller for global exponential stabilization on the $n$-dimensional sphere applies to trajectory tracking for a multi-rotor aerial vehicle, consider the following: the position dynamics of a multi-rotor vehicle can be described by

$$\dot{p} = v, \quad \dot{v} = xu + g$$

where $g$ is the acceleration of gravity, $p \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ denote the position and the velocity of the vehicle with respect to the inertial reference frame, $u$ denotes the magnitude of the thrust and $x \in \mathbb{S}^2$ denotes the direction of the thrust (cf. Hamel, Mahony, Lozano, and Ostrowski (2002)). Given a reference trajectory with acceleration $\ddot{p}_d$ and a control law $w : \mathbb{R}^3 \to \mathbb{R}^3$ that exponentially stabilizes the double integrator, if the controller for $x$ exponentially stabilizes the commanded thrust direction, given by $\rho(\widetilde{p}, \widetilde{v}, \ddot{p}_d) := \frac{w(\widetilde{p},\widetilde{v})-g+\ddot{p}_d}{|w(\widetilde{p},\widetilde{v})-g+\ddot{p}_d|}$ for each $(\widetilde{p}, \widetilde{v}, \ddot{p}_d) \in \{(\widetilde{p}, \widetilde{v}, \ddot{p}_d) \in \mathbb{R}^9 : w(\widetilde{p}, \widetilde{v}) - g + \ddot{p}_d \neq 0\}$ where $\widetilde{p}$ and $\widetilde{v}$ denote the position and velocity tracking errors, respectively, then exponential tracking is attained. This, however, cannot be achieved through continuous feedback because it is not possible to globally asymptotically

☆ The material in this paper was partially presented at the 2016 American Control Conference, July 6–8, 2016, Boston, MA, USA and the 2015 American Control Conference, July 1–3, 2015, Chicago, IL, USA. This paper was recommended for publication in revised form by Associate Editor Romain Postoyan under the direction of Editor Daniel Liberzon.

* Corresponding author.
*E-mail addresses:* pcasau@isr.ist.utl.pt (P. Casau), mayhew@ece.ucsb.edu (C.G. Mayhew), ricardo@ucsc.edu (R.G. Sanfelice), cjs@isr.ist.utl.pt (C. Silvestre).

1 Pedro Casau is also with the Institute for Systems and Robotics at Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal.

2 C. Silvestre is with the Department of Electrical and Computer Engineering of the Faculty of Science and Technology of the University of Macau, Macau, China, on leave from Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal.

https://doi.org/10.1016/j.automatica.2019.108534

stabilize a given setpoint on a compact manifold by means of continuous feedback (cf. Bhat and Bernstein (2000)). Moreover, if a dynamical system cannot be globally asymptotically stabilized through continuous feedback, it cannot be robustly stabilized by discontinuous feedback either, as shown in Mayhew and Teel (2011). Overcoming these limitations is particularly important for multi-rotor aerial vehicles due to their popularity and the wide range of applications in which they are used, such as surveillance, tracking, search and rescue, infrastructure inspection, agriculture and disaster mitigation (see e.g Augugliaro, Schoellig, and D'Andrea (2013), Fink, Michael, Kim, and Kumar (2011), Floreano and Wood (2015), Jiang and Kumar (2013), Liang, Fang, Sun, and Lin (2018), Lupashin et al. (2014) and Mellinger, Michael, and Kumar (2012)). In this paper, we present a hybrid controller that not only provides global exponential stability, but also confers a quantifiable robustness margin to perturbations.

Recent developments on hybrid control theory overcame the topological obstructions to global stabilization on compact manifolds with the introduction of synergistic potential functions in Mayhew and Teel (2010). A potential function on a compact manifold is a continuously differentiable function that is positive definite relative to a given point, thus it induces a gradient vector field which asymptotically stabilizes the given reference from every initial condition except a set of measure zero. To see this, consider the closed-loop system resulting from the interconnection of the gradient-based feedback law of a potential function $h_r$ that is positive definite relative to $r \in \mathbb{S}^n$ and (1), given by:

$$\dot{x} = -\Pi(x)\nabla h_r(x). \tag{2}$$

The equilibrium points of (2) are the critical points of $h_r(x)$, denoted by

$$\operatorname{crit} h_r := \{x \in \mathbb{S}^n : \Pi(x)\nabla h_r(x) = 0\}, \tag{3}$$

which correspond to the set of points $x$ where $\nabla h_r(x)$ is orthogonal to $\mathsf{T}_x\mathbb{S}^n$. Since the set (3) includes the maximum and the minimum of $h_r$ on $\mathbb{S}^n$, it follows that $r \in \mathbb{S}^n$ is not globally asymptotically stable for (2). On the other hand, synergistic potential functions are collections of potential functions that enable a controller to achieve robust global asymptotic stabilization of the given setpoint, because, at the undesired equilibria, there exists another function in the collection with a lower value that we can switch to.

The concept of synergistic potential functions has been introduced to address the problem of stabilizing a three-dimensional pendulum in Mayhew and Teel (2010) and later used in full attitude stabilization in Mayhew and Teel (2013b) and Lee (2015); attitude synchronization (Mayhew, Sanfelice, Sheng, Arcak, & Teel, 2012); partial attitude stabilization (cf. Mayhew and Teel (2013a)), and stabilization by hybrid backstepping (cf. Mayhew, Sanfelice, and Teel (2011b)). More recently, the interest in this control technique has spawned the design of new synergistic potential functions on SO(3), such as the ones by Berkane et al. (2017) and Berkane and Tayebi (2015, 2017). It was shown by the authors in Casau, Sanfelice, Cunha, Cabecinhas and Silvestre (2015) that synergistic potential functions can be used for global asymptotic stabilization of a reference trajectory for a multi-rotor aerial vehicle, but the extent to which exponential stabilization is possible was not addressed.

### 1.2. Contributions

The contributions in this paper are as follows. *Extension to the concept of synergistic potential functions:* in Section 3, we show that the existence of a centrally synergistic potential function $V : \mathbb{S}^n \times \mathcal{Q} \to \mathbb{R}_{\geq 0}$, where $\mathcal{Q}$ is a compact set, induces a gradient-based control law that renders

$$\mathcal{A} := \{(x, y) \in \mathbb{S}^n \times \mathcal{Q} : x = r\} \tag{4}$$

globally asymptotically stable for the closed-loop system. This nomenclature is inherited from Mayhew and Teel (2013a), where synergistic potential functions satisfying $V(r, y) = 0$ for all $y \in \mathcal{Q}$ are said to be central because they share are a common minimum at $r$. In this paper, we extend the previous notion of centrally synergistic potential functions, because we consider that $\mathcal{Q}$ is compact rather than finite, which adds flexibility to the design of synergistic potential functions. The proposed controller is significantly different from the one in Mayhew and Teel (2013b) because the focus is on global exponential stabilization of $\mathbb{S}^n$ rather than SO(3) and it further expands the work in Mayhew and Teel (2010) from global asymptotic stabilization on $\mathbb{S}^2$ to global exponential stabilization on $\mathbb{S}^n$. Interestingly, the controller that we propose may be used for global stabilization on $\mathbb{S}^3$ which is the universal cover of SO(3). However, the resulting controller would be more complex than that of Mayhew, Sanfelice, and Teel (2011a) if used for rigid-body stabilization, because it would not take advantage of the fact that $\mathbb{S}^3$ is a double cover of SO(3). *Global exponential stability of a setpoint on* $\mathbb{S}^n$*:* in Theorem 2, we show that, if $V$ is bounded from above and below by a polynomial function of the distance to $\mathcal{A}$ and if $V$ converges exponentially fast to 0, then $\mathcal{A}$ is globally exponentially stable for the closed-loop system, in the sense that, for all initial conditions, the state of the system converges exponentially to $\mathcal{A}$. This is different from the controllers proposed in Berkane et al. (2017) and Lee (2015), because these address the problem of global exponential stabilization on SO(3) rather than $\mathbb{S}^n$. *Optimal switching:* in Section 3.1, we construct a synergistic potential function on the $n$-dimensional sphere that meets the requirements for exponential stability and has an optimal switching law, in the sense that it guarantees that solutions to the closed-loop system follow geodesics whenever a jump of the hybrid controller is triggered. *Robustness to perturbations:* the proposed hybrid controller satisfies the so-called hybrid basic conditions, therefore it is endowed with nominal robustness properties that are outlined in Goebel, Sanfelice, and Teel (2012). In addition, the switching rule introduces a hysteresis gap that prevents chattering. *Saturated thrust feedback for exponential tracking of a reference trajectory for a multi-rotor aerial vehicle:* in Section 4, we employ the hybrid controller for global exponential stabilization on $\mathbb{S}^n$ in trajectory tracking for a multi-rotor aerial vehicle. We show that, for each compact set of initial position and velocity tracking errors and for all initial orientations, the reference trajectory is exponentially stable. This is particularly difficult, because in addition to the topological constraints to global attitude stablilization, multirotor aerial vehicles are subject to underactuation constraints that prevent stabilization of the vehicle when the commanded thrust is zero (cf. Lizárraga (2004)). This issue has been widely acknowledged but mostly overlooked due to its singular nature. For example, in Lee, Leok, and McClamroch (2013) this flight condition is assumed to not occur and in Hua, Hamel, Morin, and Samson (2009) the controller is turned off when the commanded thrust approaches zero. Similarly to the work of Hua, Hamel, Morin, and Samson (2015), we assume that the reference trajectory does not lead to a situation where the commanded thrust is zero, but, unlike the aforementioned approach, we explicitly build this limitation into the control design procedure so to achieve semi-global exponential stability with respect to the position and velocity errors. A video of experimental runs using this controller can be found in Casau, Mayhew, Sanfelice and Silvestre (2017).

The paper is organized as follows: in Section 2, we present the notation and the framework of hybrid dynamical systems that is used in this paper. In Section 3, we develop the notions of synergistic potential functions on the $n$-dimensional sphere. In Section 4, we apply the given controllers to the tracking of a reference trajectory for a vectored-thrust vehicle, and in Section 5, we present the conclusions of this work. In Casau, Mayhew, Sanfelice and Silvestre (2015) we reported the results that are presented in Section 3 without the proofs. In Casau, Mayhew, Sanfelice, and Silvestre (2016), we presented some preliminary results on work reported in Section 4 without the constructive controller synthesis that is presented in this paper for the quadrotor application.

## 2. Preliminaries

The symbol $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of natural numbers and zero, $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space equipped with the norm $|x| := \sqrt{\langle x, x\rangle}$ for each $x \in \mathbb{R}^n$, where $\langle u, v\rangle := u^\top v$ for each $u, v \in \mathbb{R}^n$. The canonical basis for $\mathbb{R}^n$ is denoted by $\{e_i\}_{1\leq i\leq n} \subset \mathbb{R}^n$ and $c + r\mathbb{B} := \{x \in \mathbb{R}^n : |x - c| \leq r\}$. If a real symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive (negative) definite, we write $A \in \mathbb{S}^n_{>0}$ ($A \in \mathbb{S}^n_{<0}$) or $A \succ 0$ ($A \prec 0$) if the dimensions can be inferred from context. If a real symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive (negative) semidefinite, we write $A \in \mathbb{S}^n_{\geq 0}$ ($A \in \mathbb{S}^n_{\leq 0}$) or $A \succeq 0$ ($A \preceq 0$) if the dimensions can be inferred from context. Given $A \in \mathbb{R}^{m\times n}$, $\sigma_{\max}(A)$ denotes the maximum singular value of $A$. The gradient of a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is given by $\nabla V(x) := \begin{bmatrix}\frac{\partial V}{\partial x_1}(x) & \dots & \frac{\partial V}{\partial x_n}(x)\end{bmatrix}$ for each $x = (x_1, \dots, x_n) \equiv \begin{bmatrix}x_1 & \dots & x_n\end{bmatrix}^\top \in \mathbb{R}^n$. The derivative of a differentiable matrix function with matrix arguments $F : \mathbb{R}^{m\times n} \to \mathbb{R}^{k\times \ell}$ is given by $\mathcal{D}_X(F(X)) := \partial \mathrm{vec}(F(X)) / \partial \mathrm{vec}(X)^\top$ for each $X \in \mathbb{R}^{m\times n}$, where $\mathrm{vec}(X) := \begin{bmatrix}e_1^\top X^\top & \dots & e_n^\top X^\top\end{bmatrix}^\top$. The domain of a set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is given by $\mathrm{dom}\, M := \{x \in \mathbb{R}^n : M(x) \neq \emptyset\}$. The range of $M$ is the set $\mathrm{rge}\, M := \{y \in \mathbb{R}^n : \exists x \in \mathbb{R}^m \text{ such that } y \in M(x)\}$. A hybrid system $\mathcal{H}$ defined on $\mathbb{R}^n$ can be represented by

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases} \tag{5}$$

where $C \subset \mathbb{R}^n$ is the flow set, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the flow map, $D \subset \mathbb{R}^n$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with $D \subset \mathrm{dom}\, G$ is the jump map, as defined in Goebel et al. (2012). The maps $F$ and $G$ are set-valued maps satisfying $C \subset \mathrm{dom}\, F$ and $D \subset \mathrm{dom}\, G$, respectively. Loosely speaking, solutions to hybrid systems are hybrid arcs that are compatible with the data of the hybrid system, i.e., functions $(t, j) \mapsto \phi(t, j)$ defined on a hybrid time domain $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}_0$ that satisfy $\dot{\phi}(t, j) \in F(\phi(t, j))$ for almost all $t \geq 0$ and $\phi(t, j) \in D$, $\phi(t, j+1) \in G(\phi(t, j))$ for every $\phi(t, j)$ and $\phi(t, j+1)$ belonging to $\mathrm{dom}\, \phi$. We say that a solution $\phi$ to (5) is maximal if it cannot be extended, it is complete if its domain is unbounded, it is discrete if $\mathrm{dom}\, \phi \subset \{0\} \times \mathbb{N}$ and it is Zeno if it is complete and $\sup_t \mathrm{dom}\, \phi < +\infty$ (for a more rigorous description of solutions to hybrid systems see Goebel et al. (2012)). Under the assumption of completeness of maximal solution to $\mathcal{H}$, a compact set $\mathcal{A}$ is said to be: stable for $\mathcal{H}$, if for each $\epsilon > 0$ there exists $\delta > 0$ such that for each solution $\phi$ to $\mathcal{H}$ with $|\phi(0, 0)|_{\mathcal{A}} \leq \delta$ satisfies $|\phi(t, j)|_{\mathcal{A}} \leq \epsilon$ for each $(t, j) \in \mathrm{dom}\, \phi$; attractive for $\mathcal{H}$ if $\lim_{t+j\to\infty} |\phi(t, j)|_{\mathcal{A}} = 0$, where $|x|_{\mathcal{A}} := \min_{y\in\mathcal{A}} |x - y|$ and $|x| := \sqrt{\langle x, x\rangle}$ for each $x \in \mathbb{R}^n$. A set $\mathcal{A}$ is globally asymptotically stable for $\mathcal{H}$ if it is both globally attractive and globally stable for $\mathcal{H}$. We say that a compact set $\mathcal{A} \subset \mathbb{R}^n$ is exponentially stable in the $t$-direction from $U$ if there exists $k, \lambda > 0$ such that $\sup_t \mathrm{dom}\, \phi = \infty$ and $|\phi(t, j)|_{\mathcal{A}} \leq k \exp(-\lambda t) |\phi(0, 0)|_{\mathcal{A}}$ for each maximal solution $\phi$ to the hybrid system with $\phi(0, 0) \in U$ and $(t, j) \in \mathrm{dom}\, \phi$.

## 3. Synergistic potential functions on $\mathbb{S}^n$

In this section, we design a hybrid controller that *globally* exponentially stabilizes a given reference $r \in \mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : x^\top x = 1\}$ for the system (1) using the notion of centrally synergistic potential functions given next.

**Definition 1.** A function $h_r : \mathbb{S}^n \to \mathbb{R}$ is said to be a potential function on $\mathbb{S}^n$ relative to $r$ if it is continuously differentiable and positive definite relative to $r \in \mathbb{S}^n$, i.e., $h_r(x) \geq 0$ for each $x \in \mathbb{S}^n$ and $h_r(x) = 0$ if and only if $x = r$. We denote the collection of potential functions on $\mathbb{S}^n$ relative to $r$ by $\mathscr{P}_r$. □

**Definition 2.** Given a compact set $\mathcal{Q}$, a function $V : \mathbb{S}^n \times \mathcal{Q} \to \mathbb{R}$ is said to be a centrally synergistic potential function on $\mathbb{S}^n$ relative to $r$ if the following hold: (1) For each $y \in \mathcal{Q}$, $V^y$ is a potential function on $\mathbb{S}^n$ relative to $r$, where $V^y(x) := V(x, y)$ for each $x \in \mathbb{S}^n$; (2) There exists $\delta > 0$ such that

$$\mu(x, y) := V(x, y) - \nu(x) > \delta \tag{6}$$

for each $(x, y) \in \mathcal{E}(V) := \{(x, y) \in \mathbb{S}^n \times \mathcal{Q} : \Pi(x)\nabla V^y(x) = 0, x \neq r\}$, where

$$\nu(x) := \min_{\bar{y}\in\mathcal{Q}} V(x, \bar{y}). \tag{7}$$

We denote the collection of centrally synergistic potential functions on $\mathbb{S}^n$ relative to $r$ by $\mathcal{Q}_r$. □

Given a centrally synergistic potential function $V$, the function $(x, y) \to \mu(x, y)$ is referred to as the *synergy gap* of $V$ at $(x, y)$. The synergy gap measures the difference between the current value of the function and all other potential functions in the collection. If (6) holds, we say that $V$ has *synergy gap exceeding* $\delta$. Given $V \in \mathcal{Q}_r$, we define the hybrid controller with output $\omega \in \mathbb{R}^{n+1}$, state $y \in \mathcal{Q}$ and input $x \in \mathbb{S}^n$ as follows:

$$\left.\begin{aligned} \omega &= -\nabla V^y(x) \\ \dot{y} &= 0 \end{aligned}\right\} (x, y) \in C := \{(x, y) \in \mathcal{X} : \mu(x, y) \leq \delta\} \tag{8a}$$

$$y^+ \in \varrho(x) \qquad (x, y) \in D := \{(x, y) \in \mathcal{X} : \mu(x, y) \geq \delta\}, \tag{8b}$$

where $\mathcal{X} := \mathbb{S}^n \times \mathcal{Q}$ and

$$\varrho(x) := \arg\min_{\bar{y}\in\mathcal{Q}} V(x, \bar{y}). \tag{9}$$

The interconnection between (1) and (8) is represented by the closed-loop hybrid system $\mathcal{H} := (C, F, D, G)$, given by

$$\begin{aligned} \begin{pmatrix}\dot{x} \\ \dot{y}\end{pmatrix} &= F(x, y) = \begin{pmatrix}-\Pi(x)\nabla V^y(x) \\ 0\end{pmatrix} & (x, y) \in C \\ \begin{pmatrix}x^+ \\ y^+\end{pmatrix} &\in G(x, y) = \begin{pmatrix}x \\ \varrho(x)\end{pmatrix} & (x, y) \in D. \end{aligned} \tag{10}$$

The parameter $\delta$ that is used in the hybrid controller (8) is central to the construction of synergistic potential functions and it defines a hysteresis gap which prevents chattering. This construction is similar across many earlier works on synergistic hybrid feedback (see e.g., Mayhew and Teel (2013a, 2013b)) and it guarantees global asymptotic stability of $x = r$ for the hybrid system (10), as shown next.

**Theorem 1.** *Given $r \in \mathbb{S}^n$ and a compact set $\mathcal{Q}$, if there exists $\delta > 0$ such that $V \in \mathcal{Q}_r$ has synergy gap exceeding $\delta$, then the set $\mathcal{A}$ in (4) is globally asymptotically stable for the hybrid system $\mathcal{H}$ in (10).*

**Proof.** It follows from Proposition A.1 that Goebel et al. (2012, Assumption 6.5) is satisfied. From computations similar to Mayhew and Teel (2013a), we conclude that the growth of $V$ along solutions to (10) is bounded by $u_C, u_D$, where

$$u_C(x,y) := \begin{cases} -\left|\Pi(x)\nabla V^y(x)\right|^2 & \text{if } (x,y)\in C \\ -\infty & \text{otherwise,} \end{cases} \tag{11a}$$

$$u_D(x,y) := \begin{cases} -\delta & \text{if } (x,y)\in D \\ -\infty & \text{otherwise} \end{cases} \tag{11b}$$

for each $(x,y)\in\mathbb{S}^n\times\mathcal{Q}$. It follows from the assumption that $V$ is a synergistic potential function relative to $r$ that $\mu(x,y)>\delta$ for each $(x,y)\in\mathcal{E}(V)$, hence $u_C(x,y)<0$ for each $(x,y)\in\mathbb{S}^n\times\mathcal{Q}\setminus\mathcal{A}$. Since $u_D(x,y)<0$ for all $(x,y)\in\mathbb{S}^n\times\mathcal{Q}$, it follows from Goebel et al. (2012, Corollary 8.9) that $\mathcal{A}$ is globally pre-asymptotically stable for (10). Since $G(D)\subset C\cup D=\mathbb{S}^n\times\mathcal{Q}$, $\mathbb{S}^n\times\mathcal{Q}$ is compact and, for each $(x,y)\in C\backslash D$, $F(x,y)$ belongs to the tangent cone to $C$ at $(x,y)$, it follows from Goebel et al. (2012, Proposition 6.10) that each maximal solution to (10) is complete. □

We are also able to show that, under some additional conditions, the set (4) is globally exponentially stable in the $t$-direction for the hybrid system (10).

**Theorem 2.** *Given $r\in\mathbb{S}^n$ and a compact set $\mathcal{Q}$, if there exists $V\in\mathcal{Q}_r$ with synergy gap exceeding $\delta>0$ satisfying the following conditions*

$$\underline{\alpha}\,|x-r|^p \le V(x,y)\le \overline{\alpha}\,|x-r|^p \qquad \forall (x,y)\in C\cup D, \tag{12a}$$

$$\langle\nabla V(x,y),F(x,y)\rangle \le -\lambda V(x,y) \qquad \forall (x,y)\in C, \tag{12b}$$

*for some $p,\underline{\alpha},\overline{\alpha},\lambda>0$, then the set (4) is globally exponentially stable in the $t$-direction for the hybrid system (10).*

**Proof.** It follows from the proof of Theorem 1 that (10) satisfies (Goebel et al., 2012, Assumption 6.5), each of its maximal solutions is complete and $V$ is nonincreasing during jumps. Since $\mu(g)=0$ for each $g\in G(D)$, we have that $G(D)\cap D=\emptyset$. It follows from (12) that the conditions of Casau, Sanfelice and Silvestre (2017, Theorem 1) are satisfied, thus $\mathcal{A}$ is globally exponentially stable in the $t$-direction for the hybrid system (10). □

### 3.1. Construction of centrally synergistic potential function on $\mathbb{S}^n$

Given $r\in\mathbb{S}^n$, we construct a centrally synergistic potential function on $\mathbb{S}^n$ using the height function: $h_r(x):=1-r^\top x$ for all $x\in\mathbb{S}^n$. Let $k>0$, $\mathcal{V}:=\mathbb{S}^n\times\mathbb{S}^n\backslash\{(r,r)\}$, and define $V:\mathcal{V}\to\mathbb{R}$ for all $(x,y)\in\mathcal{V}$ as

$$V(x,y)=\frac{h_r(x)}{h_r(x)+kh_y(x)}=\frac{1-r^\top x}{1-r^\top x+k\left(1-y^\top x\right)}. \tag{13}$$

We now provide some differential properties of $V$, which follow from elementary calculation and some tedious manipulation, so they are presented without proof.

**Lemma 1.** *The function $V:\mathcal{V}\to[0,1]$ defined in (13) satisfies*

$$\nabla V^y(x)=\frac{kV(x,y)y-(1-V(x,y))r}{1-r^\top x+k\left(1-y^\top x\right)} \tag{14a}$$

$$\left|\Pi(x)\nabla V^y(x)\right|^2=\frac{2kV(x,y)\,(1-V(x,y))\left(1-r^\top y\right)}{\left(1-r^\top x+k\left(1-y^\top x\right)\right)^2} \tag{14b}$$

Given $\gamma\in\mathbb{R}$ satisfying $-1<\gamma<1$, we define the set $\mathcal{Q}\subset\mathbb{S}^n$ as

$$\mathcal{Q}=\left\{y\in\mathbb{S}^n : r^\top y\le\gamma\right\}. \tag{15}$$

The boundary of $\mathcal{Q}$ is $\partial\mathcal{Q}=\left\{y\in\mathbb{S}^n : r^\top y=\gamma\right\}$.

**Lemma 2.** *Given $r\in\mathbb{S}^n$ and $\gamma\in[-1,1)$, let $\mathcal{Q}$ be given by (15). The following hold for the function $V$ given in (13)*

$$\varrho(x)=\begin{cases} \mathcal{Q} & \text{if } x=r \\ -x & \text{if } r^\top x\ge-\gamma \\ \dfrac{\sigma\left(r^\top x\right)\Pi(x)r}{|\Pi(x)r|}+\alpha\left(r^\top x\right)x & \text{if } -1<r^\top x<-\gamma \\ \partial\mathcal{Q} & \text{if } r^\top x=-1. \end{cases} \tag{16a}$$

$$\nu(x)=\begin{cases} \dfrac{1-r^\top x}{1-r^\top x+2k} & \text{if } r^\top x\ge-\gamma \\ \dfrac{1-r^\top x}{1-r^\top x+k\left(1-\alpha\left(r^\top x\right)\right)} & \text{if } r^\top x<-\gamma \end{cases} \tag{16b}$$

*for each $x\in\mathbb{S}^n$, where $\alpha(v)=\gamma v-\sqrt{\left(1-v^2\right)\left(1-\gamma^2\right)}$ and $\sigma(v)=\gamma\sqrt{1-v^2}+v\sqrt{1-\gamma^2}$ for each $v\in[-1,1]$, and $\nu$, $\varrho$ are defined in (7) and (9), respectively.* □

**Proof.** Suppose that $x=r$. Then, according to (A.1) of Lemma A.1, $V(x,y)=0$ for all $y\in\mathcal{Q}$. Thus, any $y\in\mathcal{Q}$ attains the minimum of $y\mapsto V(r,y)=0$. If $r^\top x\ge-\gamma$, it follows that $-x\in\mathcal{Q}$. In this case, (A.2) of Lemma A.1 yields that the unconstrained minimizer of $y\mapsto V(x,y)$, which is $-x$, is also the minimizer of $y\mapsto V(x,y)$ when $y$ is constrained to $y\in\mathcal{Q}$. When $x=-r$, we have that $V(-r,y)=2/(2+k(1+r^\top y))$. Clearly, $y\mapsto V(-r,y)$ is minimized by maximizing $r^\top y$. When $y$ is constrained to $\mathcal{Q}$, the maximum value of $r^\top y$ is $\gamma$, which is attained by any $y$ satisfying $r^\top y=\gamma$, or equivalently, $y\in\partial\mathcal{Q}$. We now examine the case when $-1<r^\top x\le-\gamma$. Since $-x\notin\mathcal{Q}$, it suffices to study the solutions to the constrained minimization problem $\min\left\{y^\top x : 1-y^\top y=0, r^\top y-\gamma=0\right\}$ by means of Lagrange multipliers. □

Using (16b), we may compute (6), from which the next result follows.

**Corollary 1.** *For any given $r\in\mathbb{S}^n$ and $\gamma\in(-1,1)$, the function (13) is a centrally synergistic potential function relative to $r$ with synergy gap exceeding $\delta$, for any*

$$\delta\in\left(0,\frac{1+\gamma}{2/k+1+\gamma}\right).$$

**Proof.** This result follows from the fact that

$$\min\{\mu(x,y):(x,y)\in\operatorname{crit}V^y\backslash\{r\}\times\mathcal{Q}\} = \frac{1+\gamma}{2/k+1+\gamma}.$$

In addition to global asymptotic stability of $\mathcal{A}$ for (10), we also show below that the function (13) satisfies (12), thus it follows from Theorem 2 that global $\mathcal{A}$ is also globally exponentially stable in the $t$-direction for (10). It follows from the fact that $\mathcal{E}(V)=V^{-1}(1)$ (cf. Lemma A.1) and from $\mathcal{E}(V)\subset D$, with $D$ given in (8), that $V^*\in\mathbb{R}$, given by $V^*=\max_{(x,y)\in C}V(x,y)$, with $C$ given in (8) satisfies $0<V^*<1$. We note that $V^*$ exists since $V$ is continuous on the compact set $C$. The next theorem follows naturally from these considerations

**Theorem 3.** *The function $V\in C^1(\mathbb{S}^n\times\mathcal{Q})$ given in (13) satisfies (12) with*

$$\underline{\alpha}=(2(1+k+\sqrt{1+2k\gamma+k^2}))^{-1}, \tag{17a}$$

$$\overline{\alpha}=(2(1+k-\sqrt{1+2k\gamma+k^2}))^{-1} \tag{17b}$$

$$\lambda=\frac{2k(1-V^*)(1-\gamma)}{\left(1+k+\sqrt{1+2k\gamma+k^2}\right)^2}. \tag{17c}$$

*Therefore, the set (4) is globally exponentially stable in the $t$-direction for (10).*

**Proof.** Since $h_r(x) = 1 - r^\top x = |x-r|^2/2$ for every $x \in \mathbb{S}^n$, it follows from (A.5) that (13) satisfies (12a) with $\underline{\alpha}$ and $\overline{\alpha}$ given by (17a) and (17b), respectively. Since $r^\top y \leq \gamma$ for each $(x, y) \in \mathbb{S}^n \times \mathcal{Q}$, then $1 - r^\top y \geq 1 - \gamma$. It follows from the fact that $V(x, y) \leq V^*$ for each $(x, y) \in C$ and from Lemma A.2 that

$$\left|\Pi(x)\nabla V^y(x)\right|^2 \geq \frac{2k(1-V^*)(1-\gamma)}{\left(1+k+\sqrt{1+2k\gamma+k^2}\right)^2}V(x,y),$$

which proves that (12b) is satisfied. Global exponential stability of (4) for (10) follows from Theorem 2. □

### 3.2. Minimal geodesics

We have shown that the proposed synergistic potential function (13) ensures not only global asymptotic stability of $\mathcal{A}$ for (10), but also global exponential stability. Moreover, there is a third property of this function that is worth mentioning: for each $x_0 \in D$, the flows generated by the gradient vector field $\Pi(x)\nabla V^{\varrho(x_0)}(x)$ converge to $r$ through the minimal geodesic, i.e., the path of minimum distance between $x_0$ and $r$, as proved next. For every pair of points $p, q \in M$, where $M$ is a compact manifold, there is $c : \mathbb{R} \to M$ with $c(a) = p$ and $c(b) = q$ for some $a, b \in R$ such that the length of $c$, given by $L(c) = \int_a^b \left|\frac{dc}{dt}(\tau)\right| d\tau$, is the lowest among all other paths with endpoints $p$ and $q$ (cf. Burns and Gidea (2005)). If a path $p : [0, \infty) \to M$ is Lebesgue integrable, then its length is given by $L^\infty(p) = \int_0^\infty \left|\frac{dp}{dt}(\tau)\right| d\tau$. Considering the Riemannian manifold $\mathbb{S}^n$ with the standard Euclidean metric, its (unit-velocity) geodesics are of the form $x(t) = a\cos t + b\sin t$, for each $t \in \mathbb{R}$, with $a, b \in \mathbb{S}^n$ satisfying $\langle a, b\rangle = 0$, as shown in Petersen (2006, Example 30). Given $r \in \mathbb{S}^n$, one verifies that the minimal geodesic between $r$ and any given point $x_0 \in \mathbb{S}^n\backslash\{r, -r\}$ is given by $c(t) = x_0\cos t + \Pi(x_0)r\sin t/|\Pi(x_0)r|$, for each $t \in [0, \arccos(x_0^\top r)]$. If $x_0 = -r$, then each minimal geodesic from $x_0$ to $r$ is given by $c(t) = x_0\cos t + x^\perp\sin t$, for each $t \in [0, \pi]$, where $x^\perp \in \{y \in \mathbb{S}^n : \langle y, x_0\rangle = 0\}$. In particular, if $x_0$ is antipodal to $r$ and $n > 1$ then there are uncountably many minimal geodesics from $x_0$ to $r$.

It is possible to verify that, for every solution $(x, y)$ of (10) and each $(t, j), (t, j+1) \in \operatorname{dom}(x, y)$, the following holds: $x(t, j) = x(t, j+1)$. Therefore, $x{\downarrow_t}(t) := x(t, J(t))$, is defined for each $t \in [0, \sup_t \operatorname{dom}(x, y))$, for each solution $(x, y)$ to (10), where with $J(t) := \max\{j : (t, j) \in \operatorname{dom}(x, y)\}$. Moreover, it is absolutely continuous, hence $L^\infty(x{\downarrow_t})$ is well-defined. Choosing $V$ in (13), we show next that the solutions to the hybrid system (10) that start in the jump set have minimal length.

**Lemma 3.** *Consider the function $V \in C^1(\mathbb{S}^n \times \mathcal{Q})$ given in (13). For each solution $(x, y)$ to (10) with initial condition $(x_0, y_0) \in D$, we have that $L^\infty(x_t) = L\left(c_{x_0,r}\right)$, where $c_{x_0,r}$ denotes the minimal geodesic from $x_0$ to $r$.*

**Proof.** This result follows from the fact that the vector field $\Pi(x)\nabla V^{\varrho(x_0)}(x)$ is tangent to the geodesic connecting $r$ and $x_0$ and from Theorem 3 which shows that solutions converge to $r$. □

Note that function $h_r$ induces a gradient-vector field that generates flows along geodesics for almost all initial conditions. However, unlike the aforementioned continuous feedback law, the controller presented in this section exponentially renders a reference point globally asymptotically stable, which is not possible with continuous feedback. Moreover, if we resolve the ambiguity at $x = -r$ by means of some discontinuity, we still are left without any guarantees of robustness to small measurement noise. Since the hybrid controller presented in this section satisfies the hybrid basic conditions (Goebel et al., 2012, Assumption 6.5), the property of global asymptotic stability of $\mathcal{A}$ for (10) is endowed with robustness to small measurement noise, as discussed in Goebel et al. (2012).

## 4. Application to trajectory tracking for a quadrotor

In this section, we apply the controller proposed in Section 3 to the problem of trajectory tracking for a quadrotor vehicle, i.e., an aerial vehicle with four counter rotating rotors that are aligned with a direction which is fixed relative to the body of the vehicle, as described in Hamel et al. (2002). The dynamics of a thrust vectored vehicle such as a quadrotor can be described by

$$\dot{p} = v \tag{18a}$$

$$\dot{v} = Rru + g \tag{18b}$$

$$\dot{R} = RS(\omega) \tag{18c}$$

where $p \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ denote the position and the velocity of the vehicle with respect to the inertial reference frame (in inertial coordinates), $R \in \mathrm{SO}(3) := \{R \in \mathbb{R}^{3\times 3} : R^\top R = I_3, \det(R) = 1\}$ is the rotation matrix that maps vectors in body-fixed coordinates to inertial coordinates, $g \in \mathbb{R}^3$ represents the gravity vector and $r \in \mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ is the thrust vector in body-fixed coordinates. Furthermore, the inputs to (18) are $\omega \in \mathbb{R}^3$ and $u \in \mathbb{R}$ which represent the angular velocity in body-fixed coordinates and the magnitude of the thrust, respectively. The dynamical model (18) is a simplification of the one provided in Hamel et al. (2002) that better suits our experimental setup, since there the *Blade 200 QX* quadrotor that is used in the experiments has an embedded controller that tracks angular velocity and thrust commands. Furthermore, we assume that the reference trajectory satisfies the following assumption.

**Assumption 1.** The reference trajectory $t \mapsto p_d(t)$ is defined for each $t \geq 0$ and there exist $M_2 \in (0, |g|)$ and $M_3 > 0$ such that $|\ddot{p}_d(t)| \leq M_2$ and $\left|p_d^{(3)}(t)\right| \leq M_3$ for all $t \geq 0$.

Given a path that satisfies Assumption 1, we define the tracking errors as $\widetilde{p} := p - p_d$ and $\widetilde{v} := v - \dot{p}_d$, whose dynamics can be derived from (18) and are given by:

$$\dot{\widetilde{p}} = \widetilde{v}, \quad \dot{\widetilde{v}} = Rru + g - \ddot{p}_d. \tag{19}$$

Given $w : \mathbb{R}^6 \to \mathbb{R}^3$, we define

$$\rho(z) := \frac{w(\widetilde{p},\widetilde{v}) - g + \ddot{p}_d}{|w(\widetilde{p},\widetilde{v}) - g + \ddot{p}_d|} \tag{20a}$$

$$\kappa_u(z, R) := r^\top R^\top(w(\widetilde{p},\widetilde{v}) - g + \ddot{p}_d) \tag{20b}$$

for each $z := (\ddot{p}_d, \widetilde{p}, \widetilde{v}) \in M_2\mathbb{B} \times \mathbb{R}^6$ and $R \in \mathrm{SO}(3)$. Note that, if $Rr = \rho(z)$ and $u = \kappa_u(z, R)$, we obtain

$$\dot{\widetilde{p}} = \widetilde{v}, \quad \dot{\widetilde{v}} = w(\widetilde{p},\widetilde{v}) \tag{21}$$

from (19), provided that

$$w(\widetilde{p},\widetilde{v}) - g + \ddot{p}_d \neq 0. \tag{22}$$

On the other hand, if $Rr \neq \rho(z)$, then $\kappa_u(z, R)$ is the solution to the least-squares problem:

$$\min\{|Rru + g - \ddot{p}_d - w(\widetilde{p},\widetilde{v})|^2 : u \in \mathbb{R}\}.$$

The mismatch between $Rr$ and $\rho(z)$ may lead to an increase in the position and velocity tracking errors until the thrust vector $Rr$ is aligned with $\rho(z)$. The controller design is a two-step process, where we start by designing a feedback law $(\widetilde{p},\widetilde{v}) \mapsto w(\widetilde{p},\widetilde{v})$ that exponentially stabilizes the origin of (21) and then we design a partial attitude tracking controller that exponentially stabilizes $\rho(z)$ for the dynamics of the thrust vector $Rr$.

### 4.1. Controller for the position subsystem

We show next that it is possible to satisfy (22) and exponentially stabilize the origin of (21) from an arbitrary compact set $U$ using a continuously differentiable saturated linear feedback law

$$w(\widetilde{p},\widetilde{v}) := \mathrm{Sat}_b\left(K\begin{bmatrix}\widetilde{p}^\top & \widetilde{v}^\top\end{bmatrix}^\top\right) \tag{23}$$

satisfying $\mathrm{Sat}_b(x) = x$ for all $|x| \leq b$ and $|\mathrm{Sat}_b(x)| < |g| - M_2$ for each $x \in \mathbb{R}^3$ where $b \in (0, |g| - M_2)$. Defining $\Omega_P(\ell) := \{x \in \mathbb{R}^n : x^\top P x \leq \ell\}$ for $P \in \mathbb{R}^{n\times n}$ and $\ell > 0$, if we select $(H, \ell_H) \in \mathbb{S}^6_{>0} \times \mathbb{R}_{>0}$ such that $\Omega_H(\ell_H)$ is a bounding ellipsoid for $U$, then we guarantee that the bounds $\pm b$ of the saturation function (23) are not reached if there exist $(P, \ell_P) \in \mathbb{S}^6_{>0} \times \mathbb{R}_{>0}$ and $K \in \mathbb{R}^{3\times 6}$, such that $\Omega_P(\ell_P)$ is forward invariant for every solution to (21) from $U$ and

$$U \subset \Omega_H(\ell_H) \subset \Omega_P(\ell_P) \subset \Omega_{K^\top K}(b^2). \tag{24}$$

For each compact set $U \subset \mathbb{R}^6$, it is possible to select controller parameters that satisfy (24) as well as

$$(A + BK)^\top P + P(A + BK) \preceq -\widehat{Q} - K^\top \widehat{R} K \tag{25}$$

where

$$A := \begin{bmatrix}0 & I_3\\ 0 & 0\end{bmatrix}, \quad B := \begin{bmatrix}0\\ I_3\end{bmatrix}.$$

thus guaranteeing also the exponential stability of the origin of (21) from $U$, as stated in the proposition below.

**Proposition 1.** *For each compact set $U \subset \mathbb{R}^6$, $b > 0$, $\ell_P > 0$, $\ell_H > 0$, $\widehat{R} \in \mathbb{S}^3_{>0}$ and $H \in \mathbb{S}^6_{>0}$, there exist $K \in \mathbb{R}^{3\times 6}$ and $\widehat{Q}, P \in \mathbb{S}^6_{>0}$ such that the conditions (24) and (25) are satisfied.*

**Proof.** This result follows from the application of Saberi, Stoorvogel, and Sannuti (2012, Lemma 4.20). □

The previous proposition is very important to the following theorem, which constitutes the main result of this section.

**Theorem 4.** *For each $b \in (0, |g| - M_2)$, and each compact set $U \subset \mathbb{R}^6$, there exists $K \in \mathbb{R}^{3\times 6}$ such that the origin of the closed-loop system resulting from the interconnection between (21) and (23) is exponentially stable from $U$. Moreover, each solution to (21) from $U$, denoted by $t \mapsto (\widetilde{p}, \widetilde{v})(t)$, satisfies $\left|K\begin{bmatrix}\widetilde{p}(t)^\top & \widetilde{v}(t)^\top\end{bmatrix}^\top\right| \leq b$ for each $t \geq 0$.*

**Proof.** Choosing a positive definite matrix $H \in \mathbb{S}^6_{>0}$ such that $\Omega_H(1)$ is a bounding ellipsoid for $U$, it follows from Proposition 1 that there exist $K \in \mathbb{R}^{3\times 6}$ and $\widehat{Q}, P \in \mathbb{S}^6_{>0}$ such that the conditions (24) and (25) hold for any positive definite matrix $\widehat{R} \in \mathbb{R}^{3\times 3}$ and any $\ell_P > 0$. Let $V_p(\widetilde{p},\widetilde{v}) := \begin{bmatrix}\widetilde{p}^\top & \widetilde{v}^\top\end{bmatrix} P \begin{bmatrix}\widetilde{p}^\top & \widetilde{v}^\top\end{bmatrix}^\top$, for each $(\widetilde{p},\widetilde{v}) \in \mathbb{R}^6$, which is a positive definite function relative to $\{(\widetilde{p},\widetilde{v}) \in \mathbb{R}^6 : \widetilde{p} = \widetilde{v} = 0\}$ and satisfies: $\langle \nabla V_p(\widetilde{p},\widetilde{v}), f_p(\widetilde{p},\widetilde{v})\rangle \leq -\begin{bmatrix}\widetilde{p}^\top & \widetilde{v}^\top\end{bmatrix}(\widehat{Q} + K^\top \widehat{R} K)\begin{bmatrix}\widetilde{p}^\top & \widetilde{v}^\top\end{bmatrix}^\top$, where $f_p(\widetilde{p},\widetilde{v}) := (\widetilde{v}, K\begin{bmatrix}\widetilde{p}^\top & \widetilde{v}^\top\end{bmatrix}^\top)$ for each $(\widetilde{p},\widetilde{v}) \in \Omega_P(\ell_P)$. It follows from the conditions (24), that every solution $t \mapsto (\widetilde{p},\widetilde{v})(t)$ to (21) from $U$ satisfies $\left|K\begin{bmatrix}\widetilde{p}(t)\\ \widetilde{v}(t)\end{bmatrix}\right| \leq b < |g| - M_2$ for all $t \geq 0$. □

### 4.2. Partial attitude tracking

In this section, we develop a controller for (18) that tracks a reference trajectory satisfying Assumption 1. The desired acceleration, imposed by the reference trajectory upon the vehicle, is achieved by aligning the thrust vector $Rr$ with the direction of the desired acceleration. We refer to this as partial attitude tracking because we do not control rotations around the thrust vector. For controller design purposes, let us assume the following.

**Assumption 2.** Given $\delta > 0$, there exists a function $V \in \mathcal{Q}_r$ with synergy gap exceeding $\delta$ that satisfies

$$\underline{\alpha}\,|x - r|^2 \leq V(x, y) \leq \overline{\alpha}\,|x - r|^2 \quad \forall (x, y) \in C \cup D$$
$$\left|\Pi(x)\nabla V^y(x)\right|^2 \geq \lambda_1 V(x, y) \quad \forall (x, y) \in C.$$

for some $\underline{\alpha}, \overline{\alpha}, \lambda_1 > 0$, where

$$C := \{(x, y) \in \mathbb{S}^2 \times \mathcal{Q} : \mu(x, y) \leq \delta\}, \tag{26a}$$
$$D := \{(x, y) \in \mathbb{S}^2 \times \mathcal{Q} : \mu(x, y) \geq \delta\}. \tag{26b}$$

Under Assumption 2, the application of the controller developed in Section 3 yields the closed-loop system $\mathcal{H}_1 := (C_1, F_1, D_1, G_1)$ with state $\zeta := (z, R, y) \in \mathcal{Z} := M_2\mathbb{B} \times \mathbb{R}^6 \times \mathrm{SO}(3) \times \mathcal{Q}$, given by

$$\begin{aligned} F_1(\zeta) &:= \{(F_p(p_d^{(3)}, \zeta), RS(\kappa_1(p_d^{(3)}, \zeta)), 0) : p_d^{(3)} \in M_3\mathbb{B}\}\\ &\qquad \zeta \in C_1 := \{\zeta \in \mathcal{Z} : (R^\top \rho(z), y) \in C\}\\ G_1(\zeta) &:= (z, R, \varrho(R^\top \rho(z)))\\ &\qquad \zeta \in D_1 := \{\zeta \in \mathcal{Z} : (R^\top \rho(z), y) \in D\} \end{aligned} \tag{27}$$

where $F_p(p_d^{(3)}, \zeta) := (p_d^{(3)}, \widetilde{v}, Rr\kappa_u(z, R) + g - \ddot{p}_d)$ for each $(p_d^{(3)}, \zeta) \in M_3\mathbb{B} \times \mathcal{Z}$, the inputs $u$ and $\omega$ were assigned to $\kappa_u(z, R)$ and

$$\begin{aligned}\kappa_1(p_d^{(3)}, \zeta) := S(R^\top \rho(z))\,\big(&R^\top \mathcal{D}_z\,(\rho(z))\, F_p(p_d^{(3)}, \zeta)\\ &+ (k_1 + k_p \nu^\star(z))\nabla V^y(R^\top \rho(z))\big),\end{aligned} \tag{28}$$

for each $(p_d^{(3)}, \zeta) \in M_3\mathbb{B} \times \mathcal{Z}$, respectively, with $k_1 > 0$, $k_p > 0$ and

$$\nu^\star(z) := \frac{2}{\sqrt{\underline{\alpha}}}\sigma_{\max}\left(\begin{bmatrix}0 & I_3\end{bmatrix} P^{\frac{1}{2}}\right)|w(\widetilde{p},\widetilde{v}) - g + \ddot{p}_d|\ ,$$

for each $z \in M_2\mathbb{B} \times \mathbb{R}^6$. The closed-loop system represented by $\mathcal{H}_1$ in (27) inherits the switching logic from (8) but not the same feedback law. The feedback law (28) is comprised of negative feedback of the attitude tracking error as in (8), but also a feedforward term that takes into account the fact that the reference we wish to track is not a constant. Moreover, the function $\nu^\star$ is used to increase the gain of the attitude controller as the position error increases. Given a reference trajectory satisfying Assumption 1 , compact set of initial conditions $U \subset \mathbb{R}^6$ for the position and velocity errors and a synergistic potential function satisfying Assumption 2, the controller design is as follows:

(C1) Select $H \in \mathbb{S}^6_{>0}$ such that $\Omega_H(1)$ is a bounding ellipsoid for $U$;

(C2) Select $k_p, \bar{k}_1 > 0$ so that

$$k_p \bar{k}_1 \lambda_1 > 1 \tag{29}$$

where $\lambda_1$ is given in Assumption 2;

(C3) Given $b \in (0, |g| - M_2)$ and $\widehat{R} \in \mathbb{S}^6_{>0}$, select $\ell_H = \ell_P = (1 + \bar{v})^2$, where

$$\bar{v} := \bar{k}_1\sqrt{\max_{(x,y)\in\mathbb{S}^2\times\mathcal{Q}} V(x, y)}$$

and $\widehat{Q} \in \mathbb{S}^6_{>0}$ small enough and so that the constraints (24) and (25) are feasible;

(C4) Compute the controller gain $K$ by means of the optimization problem $\min\{\mathrm{trace}(P) : (P, K) \in \chi\}$ where $\chi$ represents the constraints (24) and (25).

This controller design enables exponential stability, as proved next.

**Theorem 5.** *Let Assumptions 1 and 2 hold. For each compact set $U \subset \mathbb{R}^6$, if (C1)–(C4) are satisfied, then $\mathcal{A}_1 := \{\zeta \in \mathcal{Z} : \widetilde{p} = \widetilde{v} = 0, \rho(z) = Rr\}$ is exponentially stable in the $t$-direction from $M_2\mathbb{B} \times U \times \mathrm{SO}(3) \times \mathcal{Q}$ for the hybrid system (27).*

**Proof.** Firstly, we prove that the hybrid system (27) satisfies the hybrid basic conditions. It follows from Proposition A.1 that $\mu$ is continuous, thus both $C_1$ and $D_1$ are closed because they correspond to the inverse image of a closed set through a continuous map. It follows from $b \in (0, |g| - M_2)$ and the properties of the saturation function that $\mathcal{D}_z(\rho(z))$ is continuously differentiable for each $z \in M_2\mathbb{B} \times \mathbb{R}^6$, hence $F_1$ is outer semicontinuous and locally bounded relative to $C_1$. It follows from the outer semicontinuity of $\varrho$ that $G_1$ is outer semicontinuous relative to $D_1$ and, since $\varrho$ takes values over a compact set, $G_1$ is locally bounded relative to $D_1$. Let

$$W_1(\zeta) := \sqrt{V_p(\widetilde{p}, \widetilde{v})} + \bar{k}_1\sqrt{V(x, y)} \quad \forall \zeta \in \mathcal{Z}$$

with $P := Y^{-1}$ and $x := R^\top \rho(z)$. It follows from Assumption 2 that

$$\min\{\sqrt{\lambda_{\min}(P)}, \bar{k}_1\sqrt{\underline{\alpha}}\}\, |(\widetilde{p}, \widetilde{v}, x - r)| \le W_1(\zeta) \le \sqrt{\lambda_{\max}(P) + \bar{k}_1^2\overline{\alpha}}\, |(\widetilde{p}, \widetilde{v}, x - r)| \,.$$

for each $\zeta \in C_1 \cup D_1$. It follows from the assumptions that

$$W_1^o(\zeta; f_1) \le -\frac{\lambda_{\min}(\widehat{Q} + K^\top \widehat{R} K)}{2\sqrt{\lambda_{\max}(P)}}\sqrt{V_p(\widetilde{p}, \widetilde{v})} - \frac{\bar{k}_1 k_1 \lambda_1 \sqrt{V(x, y)}}{2} \tag{30}$$

for each $f_1 \in F_1(\zeta)$, $\zeta \in \widehat{\Omega} := \{\zeta \in \mathcal{Z} : W_1(\zeta) \le 1 + \bar{v}\}$. It follows from (24) that $(\widetilde{p}, \widetilde{v}) \in \Omega_H(1)$ implies $(\widetilde{p}, \widetilde{v}) \in \Omega_P(1)$, hence, for each solution from $M_2\mathbb{B} \times U \times \mathrm{SO}(3) \times \mathcal{Q}$, we have $V_p(\widetilde{p}(0, 0), \widetilde{v}(0, 0)) \le 1$ and, consequently, $W_1(\zeta(0, 0)) \le 1 + \bar{v}$. From (29) and (30), we have $W_1^o(\zeta; f_1) \le -\lambda W_1(\zeta)$ for each $f_1 \in F_1(\zeta)$, $\zeta \in \widehat{\Omega}$ with $\lambda := \min\left\{\frac{\lambda_{\min}(\widehat{Q}+K^\top\widehat{R}K)}{2\sqrt{\lambda_{\max}(P)}}, \frac{\bar{k}_1 k_1 \lambda_1}{2}\right\}$. Note that $V$ is a synergistic potential function relative to $r$ by assumption, thus it satisfies $V(x, g_y) \le V(x, y) - \delta$ for each $g_y \in \varrho(x)$ and $\zeta \in D_1$ by construction, and $G_1(D_1) \cap D_1 = \emptyset$ because $\mu(x, g_y) = 0 < \delta$ for each $g_y \in \varrho(x)$ for each $\zeta \in D_1$. Each solution $(t, j) \mapsto \zeta(t, j)$ to (27) from $M_2\mathbb{B} \times U \times \mathrm{SO}(3) \times \mathcal{Q}$ is such that the initial condition $\zeta(0, 0)$ belongs to the compact set $\widehat{\Omega}$. Since $W_1$ is strictly decreasing during flows and jumps of (27), it follows that $\widehat{\Omega}$ is forward invariant for each solution from $M_2\mathbb{B} \times U \times \mathrm{SO}(3) \times \mathcal{Q}$. Since $G_1(D_1) \subset C_1 \cup D_1$ and $F_1(\zeta)$ belongs to the tangent cone to $C_1$ at $\zeta$ for each $\zeta \in C_1 \backslash D_1$, it follows from Goebel et al. (2012, Proposition 6.10) that each maximal solution to $\mathcal{H}_1$ from $M_2\mathbb{B} \times U \times \mathrm{SO}(3) \times \mathcal{Q}$ is complete, hence $\mathcal{A}_1$ is exponentially stable in the $t$-direction from $M_2\mathbb{B} \times U \times \mathrm{SO}(3) \times \mathcal{Q}$ for the hybrid system (27). □

It should be pointed out that, underlying the controller design described in (C1) through (C4), there is a trade-off to be resolved: if $k_p \ll 1$, then the position controller is going to have a low gain $K$ which results in large deviations from the reference; on the other hand, if controller gain $K$ is large, the function $\nu^\star$ might increase the gain on the attitude controller beyond what is acceptable in practical terms.

## 5. Conclusions

In this paper, we have demonstrated that the existence of synergistic potential functions on $\mathbb{S}^n$ is a sufficient condition for the asymptotic stabilization of systems evolving on the $n$-dimensional sphere. Moreover, if these functions and their derivatives satisfy some additional bounds, it is possible to achieve global exponential stability. We provided a construction of synergistic potential functions which generates flows along geodesics upon switching. The proposed controller was then applied to trajectory tracking for a vectored thrust vehicle.

## Acknowledgments

This work was partially supported by the projects MYRG2018-00198-FST and MYRG2016-00097-FST of the University of Macau; by the Macau Science and Technology, Development Fund under Grant FDCT/026/2017/A1 and by Fundação para a Ciência e a Tecnologia (FCT) through Project UID/EEA/50009/2019 and grant CEECIND/04652/2017. Research by R. G. Sanfelice was partially supported by the National Science Foundation, NSF, USA Grant Nos. ECS-1710621 and CNS-1544396, by AFOSR, USA Grant Nos. FA9550-16-1-0015, FA9550-19-1-0053, and FA9550-19-1-0169, and by CITRIS, USA and the Banatao Institute, USA at the University of California.

## Appendix. Auxiliary results

**Proposition A.1.** *Let $r \in \mathbb{S}^n$ and let $\mathcal{Q}$ be a compact set. Given $V \in \mathcal{Q}_r$, the following hold:*

*(1) $r \in \mathrm{crit}(V^y)$ for each $y \in \mathcal{Q}$;*
*(2) The function (6) is continuous and the map (9) is outer semicontinuous.*

**Proof.** This result follows from the application of the Maximum Theorem in Sundaram (1996) and computations similar to Mayhew and Teel (2013a, Proposition 1). □

**Lemma A.1.** *The function $V : \mathcal{V} \to \mathbb{R}$ satisfies*

$$\begin{aligned} &\arg\min_{(x,y)\in\mathcal{V}} V(x, y) = V^{-1}(0) = \{(r, y) \in \mathcal{V}\} \\ &\arg\max_{(x,y)\in\mathcal{V}} V(x, y) = V^{-1}(1) = \{(x, x) \in \mathcal{V}\} \end{aligned} \tag{A.1}$$

*Moreover, $V(x, y)$ is positive definite on $\mathcal{V}$ relative to $\{(r, y) \in \mathcal{V}\}$ and for each $x \in \mathbb{S}^n$,*

$$\arg\min_{y\in\mathbb{S}^n} V(x, y) = \begin{cases} -x & \text{if } x \neq r \\ \mathbb{S}^n \backslash \{r\} & \text{if } x = r \end{cases} \tag{A.2}$$

$$\min_{y\in\mathbb{S}^n} V(x, y) = \frac{1 - r^\top x}{2k + 1 - r^\top x}. \tag{A.3}$$

**Proof.** Since $h_r(x) \ge 0$ for all $x \in \mathbb{S}^n$ and $h_r(x) = 0$ if and only if $x = r$, it follows that $h_r(x) + kh_y(x) > 0$ on $\mathcal{V}$. Setting $V(x, y) \le 0$, we find that $h_r(x) \le 0$, which can only be satisfied (with equality) when $x = r$. That is, $V$ attains its minimum value of zero on the set $\{(r, y) \in \mathcal{V}\}$. Similarly, $V(x, y) \ge 1$ if and only if $h_y(x) \le 0$, which is again satisfied (with equality) only when $x = y$ and thus, $V$ attains its maximum value of one on the set $\{(x, x) \in \mathcal{V}\}$. To prove (A.2) and (A.3), we note – from our previous observations – that

$$\min_{\bar{y}\in\mathbb{S}^n} V(x, \bar{y}) = \frac{1 - r^\top x}{1 - r^\top x + k\max_{\bar{y}\in\mathbb{S}^n}(1 - \bar{y}^\top x)}.$$

Since, for each $x \in \mathbb{S}^n$, $1 - y^\top x$ attains its maximal value of two at $y = -x$, this choice minimizes $y \mapsto V(x, y)$ for each $x \in \mathbb{S}^n \setminus \{r\}$;

however, when $x = r$, $V(x, y) = 0$ for any $y \in \mathbb{S}^n \backslash \{r\}$, thus proving (A.2). Eq. (A.3) follows from evaluating $V(x, y)$ with $y = -x$. □

**Corollary A.1.** *Given $y \in \mathbb{S}^n$, define $V^y : \mathbb{S}^n \to \mathbb{R}$ for each $x \in \mathbb{S}^n$ as $V^y(x) = V(x, y)$, with $V$ given by (13). Then,*

$$\operatorname{crit} V^y = \begin{cases} \{r, y\} & \text{if } r \neq y \\ \mathbb{S}^n & \text{otherwise}. \end{cases}$$

**Proof.** By definition, $x \in \operatorname{crit} V^y$ if and only if $\Pi(x)\nabla V^y(x) = 0$, or equivalently, $|\Pi(x)\nabla V^y(x)| = 0$. Noting that $V^y(x) = V(x, y)$ for each $x \in \mathbb{S}^n$, it follows that $\nabla V^y(x) = \nabla_{\tilde{r}} V(x, y)$ for each $x \in \mathbb{S}^n$. By (14b) of Lemma 1, it follows that $x \in \operatorname{crit} V^y$ if and only if

$$V^y(x)(1 - V^y(x))(1 - r^\top y) = 0. \tag{A.4}$$

Clearly, (A.4) is satisfied in three cases: $V^y(x) = 0$, $1 - V^y(x) = 0$, or $1 - r^\top y = 0$. When $1 - r^\top y = 0$, or equivalently, when $r = y$, it follows that every point in $\mathbb{S}^n$ is a critical point, since (A.4) is satisfied for every $x \in \mathbb{S}^n$. In fact, when $r = y$, $V^y(x) = 1/(1 + k)$, so that for all $x \in \mathbb{S}^n$, $\nabla V^y(x) = 0$ and obviously $\Pi(x)\nabla V^y(x) = 0$. We now examine the remaining cases. When $V^y(x) = 0$, it follows from the definition of $V^y(x) = V(x, y)$ that $1 - r^\top x = 0$, or equivalently, $x = r$. If $V^y(x) = 1$, a short calculation yields $1 - y^\top x = 0$, or equivalently, $x = y$. Thus, when $r \neq y$, $\operatorname{crit} V^y = \{r, y\}$. □

**Lemma A.2.** *The following holds*

$$0 < 1 + k - \sqrt{1 + 2k\gamma + k^2} \leq 1 - r^\top x + k(1 - y^\top x) \leq 1 + k + \sqrt{1 + 2k\gamma + k^2} \tag{A.5}$$

*for all $(x, y) \in \mathbb{S}^n \times \mathcal{Q}$.*

**Proof.** The upper and lower bounds on (A.5) follow from the solution to the optimization problem $\min/\max\{J(x, y) : 1 - y^\top y = 0, 1 - x^\top x = 0, r^\top y - \gamma = 0\}$ with $J(x, y) = 1 - r^\top x + k(1 - y^\top x)$ for each $(x, y) \in \mathbb{S}^n \times \mathcal{Q}$, by means of Lagrange multipliers. □

## References


Augugliaro, F., Schoellig, A. P., & D'Andrea, R. (2013). Dance of the flying machines: Methods for designing and executing an aerial dance choreography. *IEEE Robotics & Automation Magazine*, *20*(4), 96–104.

Berkane, S., Abdessameud, A., & Tayebi, A. (2017). Hybrid global exponential stabilization on SO(3). *Automatica*, *81*, 279–285.

Berkane, S., & Tayebi, A. (2015). On the design of synergistic potential functions on SO(3). In *Proceedings of the 54th IEEE conference on decision and control* (pp. 270–275).

Berkane, S., & Tayebi, A. (2017). Construction of synergistic potential functions on SO(3) with application to velocity-free hybrid attitude stabilization. *IEEE Transactions on Automatic Control*, *62*(1).

Bhat, S. P., & Bernstein, D. S. (2000). A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon. *Systems & Control Letters*, *39*(1), 63–70.

Brahmi, B., Saad, M., Rahman, M. H., & Ochoa-Luna, C. (2019). Cartesian trajectory tracking of a 7-DOF exoskeleton robot based on human inverse kinematics. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, *49*(3), 600–611.

Burns, K., & Gidea, M. (2005). *Differential geometry and topology : with a view to dynamical systems* (p. 389). CRC Press.

Casau, P., Mayhew, C. G., Sanfelice, R. G., & Silvestre, C. (2015). Global exponential stabilization on the n-dimensional sphere. In *Proceedings of the 2015 American control conference (ACC)* (pp. 3218–3223).

Casau, P., Mayhew, C. G., Sanfelice, R. G., & Silvestre, C. (2016). Exponential stabilization of a vectored-thrust vehicle using synergistic potential functions. In *Proceedings of the 2016 American control conference* (pp. 6042–6047).

Casau, P., Mayhew, C. G., Sanfelice, R. G., & Silvestre, C. (2017). Exponential Stabilization of a Quadrotor Vehicle, URL https://youtu.be/1DPKpxQPsMk.

Casau, P., Sanfelice, R. G., Cunha, R., Cabecinhas, D., & Silvestre, C. (2015). Robust global trajectory tracking for a class of underactuated vehicles. *Automatica*, *58*, 90–98.

Casau, P., Sanfelice, R. G., & Silvestre, C. (2017). Hybrid stabilization of linear systems with reverse polytopic input constraints. *IEEE Transactions on Automatic Control*, *62*(12), 6473–6480.

Chaturvedi, N., & McClamroch, H. (2009). Asymptotic stabilization of the inverted equilibrium manifold of the 3-D pendulum using non-smooth feedback. *IEEE Transactions on Automatic Control*, *54*(11), 2658–2662.

Chaturvedi, N., Sanyal, A., & McClamroch, N. (2011). Rigid-body attitude control. *IEEE Control Systems*, *31*(3), 30–51.

Fink, J., Michael, N., Kim, S., & Kumar, V. (2011). Planning and control for cooperative manipulation and transportation with aerial robots. *International Journal of Robotics Research*, *30*(3), 324–334.

Floreano, D., & Wood, R. J. (2015). Science, technology and the future of small autonomous drones. *Nature*, *521*(7553), 460–466.

Goebel, R., Sanfelice, R. G., & Teel, A. (2012). *Hybrid Dynamical systems: Modeling, stability, and robustness* (p. 227). Princeton University Press.

Hamel, T., Mahony, R., Lozano, R., & Ostrowski, J. (2002). Dynamic modelling and configuration stabilization for an x4-flyer. *IFAC Proceedings Volumes*, *35*(1), 217–222.

Hua, M.-D., Hamel, T., Morin, P., & Samson, C. (2009). A control approach for thrust-propelled underactuated vehicles and its application to VTOL drones. *IEEE Transactions on Automatic Control*, *54*(8), 1837–1853.

Hua, M.-D., Hamel, T., Morin, P., & Samson, C. (2015). Control of VTOL vehicles with thrust-tilting augmentation. *Automatica*, *52*, 1–7.

Jiang, Q., & Kumar, V. (2013). The inverse kinematics of cooperative transport with multiple aerial robots. *IEEE Transactions on Robotics*, *29*(1), 136–145.

Lee, T. (2015). Global exponential attitude tracking controls on SO(3). *IEEE Transactions on Automatic Control*, *60*(10), 2837–2842.

Lee, T., Leok, M., & McClamroch, N. H. (2013). Nonlinear robust tracking control of a quadrotor UAV on SE(3). *Asian Journal of Control*, *15*(2), 391–408.

Liang, X., Fang, Y., Sun, N., & Lin, H. (2018). Dynamics analysis and time-optimal motion planning for unmanned quadrotor transportation systems. *Mechatronics*, *50*, 16–29.

Lizárraga, D. A. (2004). Obstructions to the existence of universal stabilizers for smooth control systems. *Mathematics of Control, Signals, and Systems (MCSS)*, *16*(4), 255–277.

Lupashin, S., Hehn, M., Mueller, M. W., Schoellig, A. P., Sherback, M., & D'Andrea, R. (2014). A platform for aerial robotics research and demonstration: The flying machine arena. *Mechatronics*, *24*(1), 41–54.

Mahony, R., Kumar, V., & Corke, P. (2012). Multirotor aerial vehicles: Modeling, estimation, and control of quadrotor. *IEEE Robotics & Automation Magazine*, *19*(3), 20–32.

Markdahl, J., Thunberg, J., & Gonçalves, J. (2018). Almost global consensus on the n-sphere. *IEEE Transactions on Automatic Control*, *63*(6), 1664–1675.

Mayhew, C. G., Sanfelice, R. G., Sheng, J., Arcak, M., & Teel, A. R. (2012). Quaternion-based hybrid feedback for robust global attitude synchronization. *IEEE Transactions on Automatic Control*, *57*(8), 2122–2127.

Mayhew, C. G., Sanfelice, R. G., & Teel, A. (2011a). Quaternion-based hybrid control for robust global attitude tracking. *IEEE Transactions on Automatic Control*, *56*(11), 2555–2566.

Mayhew, C. G., Sanfelice, R. G., & Teel, A. R. (2011b). Synergistic Lyapunov functions and backstepping hybrid feedbacks. In *Proceedings of the 2011 American control conference* (pp. 3203–3208).

Mayhew, C. G., & Teel, A. R. (2010). Global asymptotic stabilization of the inverted equilibrium manifold of the 3-D pendulum by hybrid feedback. In *Proceedings of the 49th IEEE conference on decision and control* (pp. 679–684). IEEE.

Mayhew, C. G., & Teel, A. R. (2011). On the topological structure of attraction basins for differential inclusions. *Systems & Control Letters*, *60*(12), 1045–1050.

Mayhew, C. G., & Teel, A. R. (2013a). Global stabilization of spherical orientation by synergistic hybrid feedback with application to reduced-attitude tracking for rigid bodies. *Automatica*, *49*(7), 1945–1957.

Mayhew, C. G., & Teel, A. R. (2013b). Synergistic hybrid feedback for global rigid-body attitude tracking on SO(3). *IEEE Transactions on Automatic Control*, *58*(11), 2730–2742.

Mellinger, D., Michael, N., & Kumar, V. (2012). Trajectory generation and control for precise aggressive maneuvers with quadrotors. *International Journal of Robotics Research*, *31*(5), 664–674.

Petersen, P. (2006). *Graduate texts in mathematics*: *vol. 171*, *Riemannian geometry*. Springer New York.

Saberi, A., Stoorvogel, A., & Sannuti, P. (2012). *Systems & control: Foundations & applications*, *Internal and external stabilization of linear systems with constraints*. Boston, MA: Birkhäuser Boston.

Sundaram, R. (1996). *A first course in optimization theory* (20th ed.). New York, USA: Cambridge University Press.

Triantafyllou, P., Rovithakis, G., & Doulgeri, Z. (2018). Constrained visual servoing under uncertain dynamics. *International Journal of Control*.

Zhao, S., & Zelazo, D. (2016). Bearing rigidity and almost global bearing-only formation stabilization. *IEEE Transactions on Automatic Control*, *61*(5), 1255–1268.

**Pedro Casau** is a Researcher at the Institute for Systems and Robotics (ISR) at Instituto Superior Técnico. He received the B.Sc. in Aerospace Engineering in 2008 from Instituto Superior Técnico (IST), Lisbon, Portugal. In 2010, he received the M.Sc. in Aerospace Engineering from IST and enrolled in the Electrical and Computer Engineering Ph.D. program at the same institution which he completed with distinction and honors in 2016. While at IST, he participated on several national and international research projects on guidance, navigation and control of unmanned air vehicles (UAVs) and satellites. His current research interests include nonlinear control, hybrid control systems, vision-based control systems, controller design for autonomous air-vehicles.

**Christopher G. Mayhew** received the B.S. degree in electrical engineering from the University of California, Riverside, CA, USA, in 2005 and the M.S. and Ph.D. degrees in electrical and computer engineering from the University of California, Santa Barbara, CA, USA, in 2008 and 2010, respectively. He is currently an Engineer at SpaceX. His research interests include topological constraints in control systems, modeling, control, and simulation of hybrid and nonlinear systems, and battery management systems for automotive applications. Dr. Mayhew is a member of Tau Beta Pi.

**Ricardo G. Sanfelice** received the B.S. degree in Electronics Engineering from the Universidad de Mar del Plata, Buenos Aires, Argentina, in 2001, and the M.S. and Ph.D. degrees in Electrical and Computer Engineering from the University of California, Santa Barbara, CA, USA, in 2004 and 2007, respectively. In 2007 and 2008, he held postdoctoral positions at the Laboratory for Information and Decision Systems at the Massachusetts Institute of Technology and at the Centre Automatique et Systèmes at the École de Mines de Paris. In 2009, he joined the faculty of the Department of Aerospace and Mechanical Engineering at the University of Arizona, Tucson, AZ, USA, where he was an Assistant Professor. In 2014, he joined the University of California, Santa Cruz, CA, USA, where he is currently Professor in the Department of Electrical and Computer Engineering. Prof. Sanfelice is the recipient of the 2013 SIAM Control and Systems Theory Prize, the National Science Foundation CAREER award, the Air Force Young Investigator Research Award, and the 2010 IEEE Control Systems Magazine Outstanding Paper Award. His research interests are in modeling, stability, robust control, observer design, and simulation of nonlinear and hybrid systems with applications to power systems, aerospace, and biology.

**Carlos Silvestre** received the Licenciatura degree in Electrical Engineering from the Instituto Superior Técnico (IST) of Lisbon, Portugal, in 1987 and the M.Sc. degree in Electrical Engineering and the Ph.D. degree in Control Science from the same school in 1991 and 2000, respectively. Since 2015 he is a Professor of the Department of Electrical and Computers Engineering of the Faculty of Science and Technology of the University of Macau on leave from the Department of Electrical Engineering of the Instituto Superior Técnico. Over the past years, he has conducted research on the subjects of navigation, guidance and control of air and ocean robots. His research interests include linear and nonlinear control theory, hybrid control, sensor based control, coordinated control of multiple vehicles, networked control systems, fault detection and isolation, and fault tolerant control.