# First- and Second-Order Consensus with Constant Uniform Delays

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*Abstract*— **This paper analyzes first- and second-order consensus protocols subject to constant uniform delays, when these are applied to single and double integrator agents interacting over a directed network. The goal is to derive explicit bounds on the coupling gains of these protocols such that consensus is achieved in the presence of delays, and consequently enable designing these gains in a structured manner. To that end, the consensus protocols are first analyzed using frequency domain tools, and a necessary and sufficient bound on the coupling gain of the first-order protocol is derived by inverting a bound on the delay. However, that approach does not readily provide bounds on the coupling gains such that consensus is achieved for the second-order case. Instead, the Padé approximation of the delay is used to derive sufficient bounds on the coupling gains for that case, which is the main contribution of the paper.**

## I. Introduction

The consensus problem has been the subject of intense research interest over the past two decades, motivated by the wide variety of applications for consensus algorithms in multi-agent systems. The problem consists in driving a group of agents to an agreement over some variables of interest, and has been applied to several topics, ranging from multi-vehicle formation control [1], [2] to sensor networks [3].

The seminal work in [4] analyzed a first-order consensus protocol applied to agents with single integrator dynamics for directed networks, with both fixed and switching topologies. Later, consensus protocols for agents modeled by double integrators were introduced, for example, in [5], and more recently, consensus protocols for agents modeled as triple integrators have been analyzed, for example, in [6]. In fact, a lot of research into the consensus problem followed the seminal work in [4]. Some of this research targets consensus for higher-order dynamics, disturbances or time-delays. This work focuses on the latter. Delays appear naturally in control systems, usually deteriorate the performance of the closed-loop system, and may even cause instability. Moreover, delays can be attributed to plenty of factors, such as communications, actuators, computation, and sensors. For that reason, it is important to study the impact of delays on the system and design controllers accounting for them.

One of the contributions of the work in [4] is the analysis of first-order consensus with delays, but that analysis is restricted to undirected networks. Later, [7] extends the results to the directed case using the Lambert W function. The work in [8] follows a different direction and focuses on the special case of communication delays. In that scenario, the agent receives delayed information from a neighbor but has access to its own state immediately, leading to a time mismatch when comparing the states. The usual approach is for the agent to delay its own state information. However, there the authors study stability considering that time mismatch.

For second-order consensus, numerous works analyze convergence with delays, such as [9]–[14]. Both [9] and [10] introduce necessary and sufficient conditions for consensus of the delayed network, in the form of an upper-bound on the delay. However, the expression for this upper-bound can be further simplified. More recently, a closed-form expression for the upper-bound considering agents with general second-order dynamics is provided in [11] and the expression for double integrator modeled agents is obtained as a special case. The work in [12] approaches the delayed consensus problem using matrix-inequalities, and the work in [13] describes how delays can actually be used to achieve consensus, by replacing a derivative term with a delayed term. Also, works such as [15] and [16] focus on more general linear dynamics.

Most works in the literature describe upper-bounds on the delay such that the multi-agent system achieves consensus, for given coupling gains. In contrast with those works, an algorithm to design the coupling gains of the second-order consensus protocol, such that the multi-agent system achieves consensus for a given delay, is provided in [14]. However, no closed-form expressions for bounds on the coupling gains are provided. That is the goal of this paper.

This work analyzes both first- and second-order consensus protocols and provides closed-form expressions for bounds on the coupling gains such that consensus is achieved, when constant uniform delays influence the agents. To achieve that, both a frequency analysis and the Padé approximation are used. Overall, the main contributions of this paper are that necessary and sufficient conditions for consensus with delay are provided in the first-order case, and sufficient conditions for consensus with delay are obtained in the second-order case, in the form of bounds on the coupling gains.

The remainder of the paper is organized as follows. The notation and some relevant background are introduced in Section II. Then, the problem tackled by this paper is described in Section III, followed by the convergence analysis in Section IV. Finally, Section V presents some examples and Section VI concludes the paper.

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## II. Notation and Background

### A. Notation

The notation used throughout this paper is introduced here. Vectors are set in lower case bold and matrices in upper case bold. The set of real numbers is denoted by $\mathbb{R}$, the subsets of non-negative and positive real numbers are denoted by $\mathbb{R}_0^+$ and $\mathbb{R}^+$, respectively, and the set of complex numbers is denoted by $\mathbb{C}$. For a complex number $z \in \mathbb{C}$, $\arg(z)$ denotes its argument and $|z|$ its modulus. The $m$-dimensional Euclidean space is denoted by $\mathbb{R}^m$. The dot notation is used to define the time derivative (as in $\dot{\mathbf{x}}$), and the number of dots its order (e.g. $\ddot{\mathbf{x}}$ denotes the second time derivative). The $n \times 1$ vector of ones is denoted by $\mathbf{1}_n$, and $\mathbf{0}$ denotes a matrix of zeros whose dimensions are inferred from context.

### B. Graph Theory

A directed graph $\mathcal{G}$, usually abbreviated to digraph, consists of a pair of sets $(\mathcal{V}, \mathcal{A})$, where $\mathcal{V}$ is a non-empty finite set of vertices, and $\mathcal{A} \in \mathcal{V}^2$ is a finite set of ordered pairs of vertices, called arcs. Undirected graphs are a specific case of digraphs, and for that reason, definitions are provided for digraphs only (for a self-contained exposition on the theory of digraphs, see [17]). An arc, connecting a vertex $i$ to $j$, will be denoted by $i \to j$. Informally, for an arc $i \to j$, one says that $i$ sends information to $j$, or that $j$ "sees" $i$. The set $\mathcal{N}_i \subseteq \mathcal{V}$ of vertices seen by $i$ is called the neighborhood of $i$. If a digraph is weighted, the weight associated to an arc $i \to j$ is denoted by $k_{j\leftarrow i} \in \mathbb{R}$. When there is an arc $i \to j$, then $k_{j\leftarrow i} > 0$, and when there is no arc, $k_{j\leftarrow i} = 0$. If the digraph is not weighted, then all weights associated to arcs in the digraph are considered to be one. A directed path is an ordered sequence of arcs, connecting two distinct vertices in the digraph. Then, a digraph has a spanning tree when there is at least one vertex that has a directed path to all others. For a digraph $\mathcal{G}$ with $n$ vertices, the (directed) Laplacian matrix $\mathbf{L} = [l_{ij}] \in \mathbb{R}^{n\times n}$ is defined as $l_{ij} = -k_{i\leftarrow j}$ for $i \neq j$, and $l_{ii} = -\sum_{j\neq i} l_{ij}$. The Laplacian matrix has some relevant properties such as null row sum, meaning it has at least one null eigenvalue with eigenvector $\mathbf{1}_n$.

*Lemma 1 (see [18]):* The Laplacian $\mathbf{L}$ of a digraph $\mathcal{G}$ has a single null eigenvalue and all other eigenvalues have positive real part if and only if the digraph has a spanning tree.

### C. Stability in the Presence of a Delay

Delays are known to deteriorate the performance of a closed-loop system and can ultimately lead to instability [19]. A closed-loop system with a delay is represented in the block diagram of Fig. 1, where $G(s)$ is the open-loop transfer function and $\Delta(s) = e^{-\tau s}$ represents a delay. The following assumption will be considered throughout the paper.

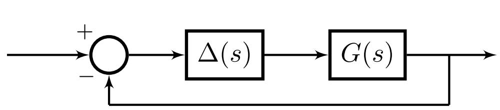



Fig. 1: Block diagram of a closed-loop system with delay.

*Assumption 1:* The SISO LTI transfer function $G(s)$ is strictly proper and can be written as $G(s) = \lambda T(s)$, where $\lambda \in \mathbb{C}$ and $T(s)$ has real coefficients. Moreover, it has a unique gain crossover frequency $\omega_0 \in \mathbb{R}^+$ for which it holds that $|\arg(T(j\omega_0))| < \pi - |\arg(\lambda)|$ and $\frac{d}{d\omega}|G(j\omega)|\big|_{\omega=\omega_0} < 0$.

Considering Assumption 1, the following result holds.

*Lemma 2:* Consider a closed-loop system with negative unitary feedback and open-loop transfer function $G(s)\Delta(s)$ where $\Delta(s) = e^{-\tau s}$. Under Assumption 1, the closed-loop system is stable if and only if $\tau < \bar{\tau}$, with $\bar{\tau} = \phi_M/\omega_0$, where $\phi_M := \pi + \arg(T(j\omega_0)) - |\arg(\lambda)|$ is the phase-margin and $\omega_0$ is the unique gain crossover frequency.

*Remark 1:* Note that the phase-margin $\phi_M$ takes the form $\phi_M = \phi_{M_r} - |\arg(\lambda)|$, where $\phi_{M_r} = \pi + \arg(T(j\omega_0))$ is the typical definition of phase margin for systems with real coefficients, i.e. $\arg(\lambda) = 0$.

The following result will be useful for the analysis.

*Lemma 3:* Let $\bar{\tau}$ be the maximum delay obtained when the open-loop transfer function is given by $\lambda T(s)\Delta(s)$, as in Lemma 2, and $\bar{\tau}_r$ the maximum delay obtained when the open-loop transfer function is given by $|\lambda|T(s)\Delta(s)$. Under Assumption 1, $\bar{\tau}_r - \bar{\tau} = \omega_0^{-1}|\arg(\lambda)|$, where $\omega_0$ is the gain crossover frequency shared by both systems.

*Proof:* Under Assumption 1, Lemma 2 is applicable. Then, since $\bar{\tau} = \omega_0^{-1}(\arg(T(j\omega_0)) + \pi - |\arg(\lambda)|)$ and $\bar{\tau}_r = \omega_0^{-1}(\arg(T(j\omega_0)) + \pi)$, the result follows. ■

### D. Padé Approximant

The delay is represented by an irrational transfer function. To use some tools from linear control theory, a rational approximation called Padé approximant to $e^{-\tau s}$ is used. The first-order Padé approximation is

$$\Delta_P(s) = \frac{2 - \tau_p s}{2 + \tau_p s}. \tag{1}$$

With this approximation, one can estimate the maximum tolerable delay by replacing the transfer function $\Delta(s)$ by $\Delta_P(s)$ in the block diagram of Fig. 1 and computing the values of $\tau_p \in \mathbb{R}^+$ for which the closed-loop system is stable.

*Lemma 4:* Let $\bar{\tau}$ be the maximum delay obtained when the open-loop transfer function is given by $G(s)\Delta(s)$, as in Lemma 2, and $\bar{\tau}_p$ the maximum delay obtained when that open-loop transfer function is approximated by $G(s)\Delta_P(s)$. Under Assumption 1, it holds that the ratio $\bar{\tau}_p/\bar{\tau}$ is given by $\bar{\tau}_p/\bar{\tau} = (\phi_M/2)^{-1}\tan(\phi_M/2)$, where $\phi_M$ is the phase-margin. Moreover, this ratio increases monotonically with $\phi_M$ for $0 < \phi_M \leq \pi/2$ and satisfies $1 < \bar{\tau}_p/\bar{\tau} \leq 4/\pi$.

*Proof:* The proof is provided in Appendix A. ■

It is worth noting that this is a relaxed approximation, in the sense that the value estimated for the maximum delay $\bar{\tau}_p$ is larger than the true value $\bar{\tau}$. For that reason, $\tau_p$ is used instead of $\tau$ in $\Delta_P(s)$, with the purpose of making a distinction between the true and the approximated values.

### E. Consensus protocols

The consensus protocols considered in the paper are introduced here. These are used for distributed coordination of a group of agents interacting over a network.

First, consider a group of $n$ agents modeled with single integrator dynamics, each described as $\dot{\beta}_i = \mu_i$, with $\beta_i, \mu_i \in \mathbb{R}$, where $\mu_i$ is the control input of the agent. The consensus protocol for this system is

$$\mu_i = -\sum_{j\in\mathcal{N}_i} k_{i\leftarrow j}\gamma_1 \left(\beta_i - \beta_j\right), \tag{2}$$

where $\gamma_1 \in \mathbb{R}^+$ is a coupling gain. The goal of the consensus protocol (2) is to guarantee that consensus is achieved, i.e., $|\beta_i - \beta_j| \to 0$ as $t \to \infty$. Note that (2) can be written in vector form using the Laplacian $\mathbf{L}$ of the digraph $\mathcal{G}$, as $\boldsymbol{\mu} = -\gamma_1 \mathbf{L}\boldsymbol{\beta}$, where $\boldsymbol{\beta} = [\beta_1 \ \cdots \ \beta_n]^\top \in \mathbb{R}^n$ and $\boldsymbol{\mu} = [\mu_1 \ \cdots \ \mu_n]^\top \in \mathbb{R}^n$. In the delay-free case, it is shown in [18] that the existence of a spanning tree on the digraph which describes the interaction topology is a necessary and sufficient condition for consensus when the agents are modeled as single integrators.

Now, consider a group of $n$ agents modeled with double integrator dynamics, each described as $\ddot{\alpha}_i = \mu_i$, with $\alpha_i, \mu_i \in \mathbb{R}$, where $\mu_i$ is the control input of the agent. The consensus protocol for this system is

$$\mu_i = -\sum_{j\in\mathcal{N}_i} k_{i\leftarrow j}\gamma_1 \left[\left(\dot{\alpha}_i - \dot{\alpha}_j\right) + \gamma_2\left(\alpha_i - \alpha_j\right)\right], \tag{3}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}^+$ are coupling gains. The goal of protocol (3) is to guarantee that $|\alpha_i - \alpha_j| \to 0$ as $t \to \infty$. Note that (3) can be written in vector form as $\boldsymbol{\mu} = -\gamma_1 \mathbf{L}\left[\dot{\boldsymbol{\alpha}} - \gamma_2 \boldsymbol{\alpha}\right]$, with $\boldsymbol{\alpha} = [\alpha_1 \ \cdots \ \alpha_n]^\top \in \mathbb{R}^n$ and $\boldsymbol{\mu} = [\mu_1 \ \cdots \ \mu_n]^\top \in \mathbb{R}^n$. In contrast with the first-order case, it is shown in [5] that the existence of a spanning tree is no longer sufficient for consensus in the delay-free case, but it was only later that necessary and sufficient conditions were presented in [10], in the form of an upper-bound on $\gamma_2/\gamma_1$.

## III. Problem Statement

The goal of this work is to design the coupling gains for the consensus protocols introduced in the previous section, in the presence of constant uniform delays. More concretely, for a given delay $\tau \in \mathbb{R}^+$, this work seeks to find bounds on the coupling gains that allow for a structured design of such gains. The vector forms of the consensus protocols (2) and (3), in the presence of a delay $\tau \in \mathbb{R}^+$ become

$$\begin{aligned} &\boldsymbol{\mu}(t) = -\gamma_1 \mathbf{L}\boldsymbol{\beta}(t-\tau), \qquad \text{and} \\ &\boldsymbol{\mu}(t) = -\gamma_1 \mathbf{L}\left[\dot{\boldsymbol{\alpha}}(t-\tau) + \gamma_2 \boldsymbol{\alpha}(t-\tau)\right], \end{aligned}$$

respectively. These delays can originate from a wide variety of factors, such as communication, actuation or computation. Note that when the $i$-th agent computes $\mu_i(t)$, the delay also influences its own data. In some scenarios (such as communication delays), that data may be available to the $i$-th agents without delay. In those cases, the data from $i$-th agent must be delayed so that the error is computed coherently. In other scenarios, the delay influences the $i$-th agent as well. Examples are: 1) an input delay — it can be associated to the communication between the controller and the actuator, or even to the time that it takes to compute the control action —, or 2) a sensing delay — it can be associated to the communication between the sensor and the controller, or to the time it takes to process some sensor data (consider for example that the sensor is a camera and some image processing needs to be performed to retrieve relevant data, or that an estimator is being used to process sensor data).

## IV. Convergence Analysis

The effect of the constant uniform delays is studied in this section. To that end, a modal decomposition that is essential to the analysis is presented next. Then, the consensuability of the delayed system is analyzed using both Lemma 2 and the Padé approximation.

### A. Modal Decomposition

The analysis of the system is simplified by performing a transformation that allows to describe it in terms of its modes, where each of these modes is associated to an eigenvalue of $\mathbf{L}$. Let $\mathbf{J}$ be the Jordan Canonical Form of $\mathbf{L}$ such that $\mathbf{L} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1}$, with $\mathbf{J} = \operatorname{blkdiag}\left(\boldsymbol{\Lambda}_1, \ldots, \boldsymbol{\Lambda}_m\right)$, where $\boldsymbol{\Lambda}_k$ is the Jordan block associated to the eigenvalue $\lambda_k \in \mathbb{C}$ of $\mathbf{L}$. Now, considering the change of coordinates $\tilde{\boldsymbol{\alpha}} = \mathbf{V}^{-1}\boldsymbol{\alpha}$ and $\tilde{\boldsymbol{\beta}} = \mathbf{V}^{-1}\boldsymbol{\beta}$, the closed-loop dynamics for the first- and second-order consensus become

$$\dot{\tilde{\boldsymbol{\beta}}}(t) = -\gamma_1 \mathbf{J}\tilde{\boldsymbol{\beta}}(t) \quad \text{and} \quad \ddot{\tilde{\boldsymbol{\alpha}}}(t) = -\gamma_1 \mathbf{J}\left[\dot{\tilde{\boldsymbol{\alpha}}}(t) + \gamma_2 \tilde{\boldsymbol{\alpha}}(t)\right],$$

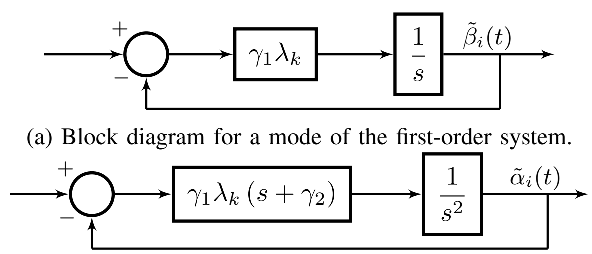
respectively. Then, for the purpose of analyzing convergence, it suffices to study the scalar systems associated to the eigenvalues of $\mathbf{L}$, namely

$$\begin{aligned} &\dot{\tilde{\beta}}_i(t) = -\gamma_1\lambda_k\tilde{\beta}_i(t) \qquad \text{and} \\ &\ddot{\tilde{\alpha}}_i(t) = -\gamma_1\lambda_k\left(\dot{\tilde{\alpha}}_i(t) + \gamma_2\tilde{\alpha}_i(t)\right), \end{aligned} \tag{4}$$

for the first- and second-order protocols, respectively. Furthermore, these modes can be described using the block diagrams depicted in Fig. 2. The following result follows immediately from the aforementioned decomposition.

*Proposition 1:* The consensus protocols (2) and (3) for single and double integrators, respectively, achieve consensus asymptotically if and only if the digraph that models the network has a spanning tree and the modes associated to the non-null eigenvalues of $\mathbf{L}$ are stable.

*Proof:* The result is a generalization of [6, Lemma 2] and the proof is similar. For that reason, it is omitted here and the reader is referred there. ∎




(a) Block diagram for a mode of the first-order system.

(b) Block diagram for a mode of the second-order system.

Fig. 2: Block diagrams for the modes defined in (4).

### B. Analysis in the Frequency Domain

The results regarding the convergence of the consensus protocols (2) and (3), obtained through direct application of Lemma 2, are discussed here. Similar results were obtained, for example, in [7] for first-order consensus, and in [10] for second-order consensus.

It is actually trivial to use the result obtained directly from Lemma 2 to design the coupling gain $\gamma_1$ for the first-order case such that, given the time-delay $\tau$, the agents achieve consensus. Since that is not discussed there, and it is the goal of this work, it is described here for completeness. For first-order consensus, the modes are characterized by the block diagram of Fig. 2a, and open-loop transfer function for a given mode associated to the eigenvalue $\lambda_k$ of $\mathbf{L}$ is $G_{1k}(s) = \lambda_k T_1(s)$, with $T_1(s) = \gamma_1/s$. It is straightforward to conclude that Assumption 1 holds for $G_{1k}(s)$, meaning that Lemma 2 is applicable and together with Proposition 1 leads to the result that follows (which had been introduced in [7]). From now on, let $\psi_k := \arg(\lambda_k)$.

*Lemma 5:* In the presence of a constant delay $\tau$, the first-order consensus protocol (2) reaches consensus asymptotically if and only if there is a spanning tree on the digraph that models the interaction topology of the agents and

$$\tau < \min_{\lambda_k \neq 0} \frac{\frac{\pi}{2} - |\psi_k|}{\gamma_1 |\lambda_k|}, \tag{5}$$

where $\lambda_k$ is the $k$-th eigenvalue of $\mathbf{L}$ and $\psi_k = \arg(\lambda_k)$.

Note that it is straightforward to use (5) to determine the maximum delay $\bar{\tau}$ for a given coupling gain $\gamma_1$. However, one can also use it to design the coupling gain $\gamma_1$ for a given delay $\tau$, by solving for $\gamma_1$ in (5).

In contrast with the first-order case, it turns out that the result obtained through direct application of Lemma 2 for second-order consensus (which can be derived from [10, Theorem 2] by rewriting the expression to obtain an explicit bound) does not readily provide bounds on the coupling gains such that the agents achieve consensus given the time-delay. For that reason, the Padé approximation is considered in the sequel. Using that approximation, it is possible to obtain explicit bounds for the coupling gains that allow for a structured design. A drawback is that these conditions are only sufficient.

### C. Analysis using the Padé Approximation

Consider the block diagrams of Fig. 2 with a new block $\Delta(s)$, corresponding to a delay. The stability analysis is simplified by replacing $\Delta(s)$ with $\Delta_P(s)$. A disadvantage of this approach is that it yields relaxed bounds on the coupling gains, and therefore these bounds correspond to necessary but not sufficient conditions for stability. Still, sufficient bounds can be obtained using Lemma 4 to adjust the value of $\tau_p$, as will be described. This section begins with a discussion on how to address complex eigenvalues in the Laplacian, and only then the first- and second-order consensus protocols are analyzed using the Padé approximation.

*1) Dealing with complex eigenvalues in the Laplacian:* To obtain bounds on the coupling gains of the consensus protocols, and ultimately design these gains considering the time-delay, one can use, for example, the Routh-Hurwitz criterion. That is because the Padé approximation is being used to approximate the delay, resulting in linear system dynamics. However, when the eigenvalues of $\mathbf{L}$ have an imaginary part, the characteristic polynomials of the closed-loop systems have complex coefficients, and the application of the standard Routh-Hurwitz criterion is not possible. An alternative would be to use an extension of the Routh-Hurwitz criterion to polynomials with complex coefficients (see [20]). However, the computations proved to be intractable, and another approximation is proposed. Consider taking the eigenvalues of $\mathbf{L}$ and approximating them by their modulus, i.e., using $|\lambda_k|$ instead of $\lambda_k$. The error associated to this approximation is described in Lemma 3, and similarly to the Padé approximation, this also relaxes the conditions with respect to the true bounds.

To deal with the errors committed when using both approximations, i.e., when using the open-loop transfer function $|\lambda_k|T(s)\Delta_P(s)$ as an approximation for $\lambda_k T(s)\Delta(s)$, one can proceed as follows. Let $\epsilon_k := \bar{\tau}_{k_r} - \bar{\tau}_k$ be the error committed when approximating $\lambda_k$ by $|\lambda_k|$, where $\bar{\tau}_k$ and $\bar{\tau}_{k_r}$ are the maximum delays before and after the approximation. This error is described in Lemma 3. Now, suppose that the maximum delay $\bar{\tau}_p$ is obtained for $|\lambda_k|T(s)\Delta_P(s)$. Provided that the conditions in Lemma 4 hold, it follows that for $|\lambda_k|T(s)\Delta(s)$ instead of $|\lambda_k|T(s)\Delta_P(s)$, the mode associated to the eigenvalue $\lambda_k$ of $\mathbf{L}$ is stable if $\tau < \bar{\tau}_{k_r}$, with $\bar{\tau}_{k_r} = \frac{\pi}{4}\bar{\tau}_p$. Moreover, for $\lambda_k T(s)\Delta(s)$ instead of $|\lambda_k|T(s)\Delta_P(s)$ the mode associated to the eigenvalue $\lambda_k$ of $\mathbf{L}$ is stable if $\tau < \bar{\tau}_k$, with $\bar{\tau}_k = \frac{\pi}{4}\bar{\tau}_p - \epsilon_k$. Conversely, if the closed-loop system is stable with $|\lambda_k|T(s)\Delta_P(s)$ for $\tau_p = \frac{4}{\pi}(\tau + \epsilon_k)$, then it is stable with $\lambda_k T(s)\Delta(s)$ for $\tau \in \mathbb{R}^+$. Furthermore, the resulting conditions on the coupling gains obtained for the approximated open-loop transfer function are sufficient to guarantee stability of the closed-loop system with the original open-loop transfer function.

*2) First-order consensus:* The block diagram of Fig. 2a, replacing $\lambda_k$ with $|\lambda_k|$ and adding $\Delta_P(s)$ to the feedback loop is considered in the analysis. That leads to the characteristic polynomial of the closed-loop transfer function

$$p_{k_1}(s) = s^2 + (T - \tilde{\gamma}_{1_k})\, s + \tilde{\gamma}_{1_k} T,$$

with $T = 2/\tau_p$ and $\tilde{\gamma}_{1_k} = \gamma_1|\lambda_k|$. Since this is a second-order polynomial, it is Hurwitz when all coefficients have the same sign. Direct application of that condition leads to

$$\gamma_1 < \frac{2}{\tau_p |\lambda_k|}. \tag{6}$$

It is straightforward to compute the gain crossover frequency of the mode associated to the eigenvalue $\lambda_k$ of $\mathbf{L}$, given as $\omega_{0_k} = \gamma_1|\lambda_k|$, meaning that $\epsilon_k = |\psi_k|/(\gamma_1|\lambda_k|)$, as described in Lemma 3. Replacing $\tau_p$ with $\frac{4}{\pi}(\tau + \epsilon_k)$ in (6) and solving for $\gamma_1$, leads to the exact bound described in Lemma 5, thus validating this approach. The reason why it was possible to obtain the exact bound in the first-order case is because the errors committed in both approximations are known exactly (in this case $\phi_M = \pi/2$ and the upper-bound on $\bar{\tau}_p/\bar{\tau}$, provided in Lemma 4, is tight). That will no longer be true for second-order consensus.

*3) Second-order Consensus:* The block diagram of Fig. 2b, replacing $\lambda_k$ with $|\lambda_k|$ and adding $\Delta_P(s)$ to the feedback loop is considered in the analysis. Then, the characteristic polynomial of the closed-loop system becomes

$$p_{k_2}(s) = s^3 + (T - \tilde{\gamma}_{1k})\, s^2 + \tilde{\gamma}_{1k}\,(T - \gamma_2)\, s + \tilde{\gamma}_{1k}\gamma_2 T,$$

where $\tilde{\gamma}_{1k} = |\lambda_k|\gamma_1$ and $T = 2/\tau_p$.

*Lemma 6:* A third-order polynomial, of the form

$$p_2(s) = s^3 + (T - k_1)\, s^2 + k_1\,(T - k_2)\, s + k_1 k_2 T,$$

has all roots in the open left half-plane if and only if

$$\begin{cases} k_1 < T \\ k_2 < T\left(1 - \dfrac{T}{2T - k_1}\right) \end{cases},$$

with $k_1, k_2, T \in \mathbb{R}^+$.

Lemma 6 can be applied to $p_{k_2}(s)$ to obtain bounds on the coupling gains. However, to use $\tau_p = \frac{4}{\pi}(\tau + \epsilon_k)$ in those bounds and ultimately obtain sufficient conditions, it is necessary to verify that Lemma 4 is applicable. That is done by noting that for $G_{2k}(s)$ with $\lambda_k$ replaced with $|\lambda_k|$, it holds that $\phi_M = \arg(\gamma_2 + j\omega) < \pi/2$, for all $\omega \in \mathbb{R}^+$ and all $\lambda_k \neq 0$, because $\gamma_1, \gamma_2 \in \mathbb{R}^+$. Therefore, Lemma 4 can be applied to conclude that $\bar{\tau}_p/\tau < 4/\pi$.

It is still necessary to compute the gain crossover frequency $\omega_{0_k}$, because $\epsilon_k = |\psi_k|/\omega_{0_k}$. That leads to

$$\omega_{0_k} = |\lambda_k|\gamma_1 r\left(\frac{\gamma_2}{\gamma_1|\lambda_k|}\right), \tag{7}$$

with $r(x) := \sqrt{\frac{1}{2}\left(1 + \sqrt{1 + 4x^2}\right)}$. However, it turns out that knowing the exact expression for $\epsilon_k$ is not very useful in this case. That is because $\epsilon_k$ depends on the crossover frequency, which in turn depends on both $\gamma_1$ and $\gamma_2$. Following the same rationale used for the first-order case would lead to bounds for $\gamma_1$ and $\gamma_2$ which would depend on $\gamma_1$ and $\gamma_2$ themselves. This would lead to a somewhat recursive design which is not the purpose of this paper. To overcome this issue, an upper-bound on $\epsilon_k$ is used, which is obtained through a lower-bound on the crossover frequency. It follows from (7) that $\omega_{0_k} > \gamma_1|\lambda_k|$, which leads to $\epsilon_k < |\psi_k|/\left(\gamma_1|\lambda_k|\right)$. Finally, the following is the main result of the paper.

*Theorem 1:* In the presence of a constant delay $\tau$, the second-order consensus protocol (3) reaches consensus asymptotically if there is a spanning tree on the digraph that models the interaction topology of the agents and

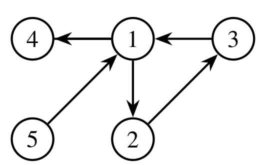
$$\begin{cases} \gamma_1 < \min\limits_{\lambda_k \neq 0} \dfrac{\frac{\pi}{2} - |\psi_k|}{|\lambda_k|\tau} \\ \gamma_2 < \min\limits_{\lambda_k \neq 0} \dfrac{\frac{\pi}{2}\gamma_1|\lambda_k|}{\tau\gamma_1|\lambda_k| + |\psi_k|}\left(1 - \dfrac{\frac{\pi}{2}}{\pi - (|\psi_k| + \gamma_1|\lambda_k|\tau)}\right) \end{cases},$$

where $\lambda_k$ is the $k$-th eigenvalue of $\mathbf{L}$ and $\psi_k = \arg(\lambda_k)$.

*Proof:* Lemma 6 can be used to obtain bounds on the coupling gains for which the polynomial $p_{k_2}(s)$ is Hurwitz. However, to obtain sufficient conditions for the original system, one must use $\tau_p = \frac{4}{\pi}(\tau + \epsilon_k)$. Since the exact value of $\epsilon_k$ cannot be used, the upper-bound is used instead. In other words, $\tau_p = \frac{4}{\pi}\left(\tau + |\psi_k|/\left(\gamma_1|\lambda_k|\right)\right)$ can be used in the bounds on the coupling gains obtained through Lemma 6 to retrieve sufficient conditions for the stability of the mode associated to the eigenvalue $\lambda_k$. Note that the bound on $\gamma_1$ would depend on $\gamma_1$ itself. However, $\gamma_1$ can be isolated as in the first-order case. Finally, the conditions follow from Proposition 1, by noting that all modes such that $\lambda_k \neq 0$ must be stable. ■

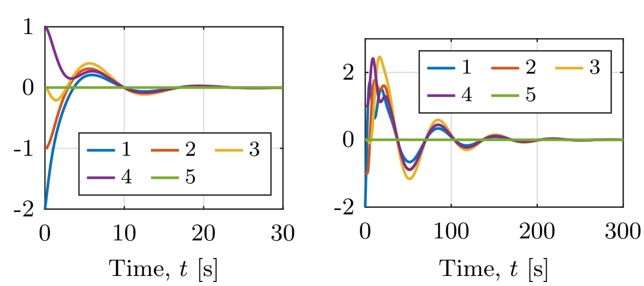
## V. Illustrative Examples

Suppose that one wishes to design the consensus protocol for a group of double integrator modeled agents that communicate over a network modeled by the digraph of Fig. 3. First, note that the digraph contains a spanning tree. The eigenvalues of the Laplacian are $\lambda_1 = 0$, $\lambda_2 = 0.2451$, $\lambda_3 = 1$ and $\lambda_{4,5} = 1.8774 \pm 0.7449j$. Then, Theorem 1 can be used to design the coupling gains. Suppose that there is a delay of $\tau = 0.12\,\mathrm{s}$. One starts by computing the bound on $\gamma_1$, for which it is obtained that it must be $\gamma_1 < \gamma_{1c}$ with $\gamma_{1c} = 4.9225$. The critical condition occurs for $\lambda_{4,5}$, since these eigenvalues have both the highest modulus and non-null phase. Suppose one chooses $\gamma_1 = \gamma_{1c}/2$. Then, the bound on $\gamma_2$ leads to $\gamma_2 < \gamma_{2c}$ with $\gamma_{2c} = 5.4299$. Choosing $\gamma_2 = 2\gamma_{2c}/3$ leads to the time response of Fig. 4a, where clearly the agents reach consensus. Now, suppose that $\tau = 1.2\,\mathrm{s}$. Using the same process to compute again $\gamma_{1c}$ and choosing $\gamma_1 = \gamma_{1c}/2$, and then computing $\gamma_{2c}$ and using $\gamma_2 = 2\gamma_{2c}/3$, leads to the time response of Fig. 4b. Clearly the agents reach consensus but convergence is slower. That is expected, since the delay imposes a limit on the bandwidth of the system. Surely, with a bigger delay the coupling gains must be decreased in order to keep the system stable.




$$\mathbf{L} = \begin{bmatrix} 2 & 0 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 3: Digraph and associated matrix $\mathbf{L}$ for the examples.




(a) Time response for $\tau = 0.12$ (b) Time response for $\tau = 1.2$

Fig. 4: Time response for the two different delays.

## VI. Conclusion

This paper set out to provide a method to design the coupling gains for first- and second-order consensus when there is a delay in the feedback loop. To achieve that goal, a frequency analysis was conducted at first. That analysis led to necessary and sufficient conditions for consensus with time-delays, in the form of a bound on the coupling gain. However, the same method could not be successfully applied to the second-order case, and the Padé approximation of the delay was used instead. With this approach, it was possible to replicate the result obtained for first-order consensus, and also obtain novel sufficient conditions for consensus with time-delays, in the form of bounds on the coupling gains. These conditions enable a structured design of the coupling gains. To conclude the paper, the results were illustrated in an example. A future research direction is to extend the analysis to third-order consensus, which is relevant, for example, in formation control.

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## Appendix

### A. Proof of Lemma 4

If the closed-loop system is stable without delay, the same argument used in the proof of Lemma 2 can be used to conclude that the system is stable until the poles first hit the imaginary axis. That happens when $1 + \Delta(j\omega)G(j\omega) = 0$ for some $\omega \in \mathbb{R}^+$ (it was shown in the proof of Lemma 2 that it is possible to focus on $\omega > 0$ only). More concretely, the poles first hit the imaginary axis when

$$\arg(\Delta(j\omega_0)) = -\Big[\arg(T(j\omega_0)) + (2k+1)\pi - |\arg(\lambda)|\Big],$$

where $\omega_0 \in \mathbb{R}^+$ is the unique crossover frequency. Since it must be $\arg(\Delta(j\omega_0)) < 0$ and from Assumption 1 it holds that $-\pi < \arg(G(j\omega)) < \pi$, then $k \in \mathbb{Z}$ can be set to zero to obtain $\arg(\Delta(j\omega_0)) = -\phi_M$. This in turn leads to the expression for $\bar{\tau}$ presented in Lemma 2. If one replaces $\Delta(s)$ with $\Delta_P(s)$, the same rationale applies. For the Padé approximation, it holds that

$$\arg(\Delta_P(j\omega)) = \arg\left(\frac{2 - j\omega\tau_p}{2 + j\omega\tau_p}\right) = -2\arctan\left(\frac{\omega\tau_p}{2}\right),$$

meaning that the value of $\tau_p$ for the approximation such that $\arg(\Delta_P(j\omega_0)) = -\phi_M$ (i.e., $\tau_p$ that first sets poles on the imaginary axis) is

$$\tau_p = \bar{\tau}_p := \frac{2\tan(\phi_M/2)}{\omega_0} \implies \frac{\bar{\tau}_p}{\bar{\tau}} = \frac{\tan(\phi_M/2)}{\phi_M/2}$$

This ratio increases monotonically with $\phi_M$, as one would expect. This means that, if $0 < \phi_M \le \pi/2$, it holds that $1 < \bar{\tau}_p/\bar{\tau} \le (\pi/4)^{-1}\tan(\pi/4) = 4/\pi$.